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Qualitative Theory of Dynamical Systems

Infinitely Many Periodic Orbits for the Octahedral 7-Body Problem

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Abstract. We prove the existence of infinitely many symmetric periodic orbits for a regularized octahedral 7-body problem with six small masses placed at the vertices of an octahedron centered in the seventh mass. The main tools for proving the existence of such periodic orbits is the analytic continuation method together with the symmetry of the problem.

Keywords. Continuation method, symmetric periodic orbits, 7-body problem.

1. Introduction

In this paper we consider a particular case of the spatial 7-body problem defined as follows. We consider a mass $m_0 = 1$ located at the origin of coordinates with zero initial velocity, two small masses $m_1 = m_2 = \mu\nu_1$ with initial positions and velocities on the x-axis symmetric with respect to the origin, two small masses $m_3 = m_4 = \mu\nu_2$ with initial positions and velocities on the y-axis symmetric with respect to the origin, and finally two small masses $m_5 = m_6 = \mu\nu_3$ with initial positions and velocities on the z-axis symmetric with respect to the origin (see Figure 1). Our 7-body problem consists of describing the motion of the seven masses under their mutual Newtonian gravitational attraction. Due to the symmetry of the initial conditions and velocities, the six small bodies are located at any time in the vertices of an octahedron with center at m_0, and the mass m_0 remains in rest at the origin. We call the *octahedral* 7*-body problem* the study of the motion of this 7-body problem.

Although this is a 7-body problem it can be formulated as a Hamiltonian system of three degrees of freedom, one is the distance $x \geq 0$ of m_1 to the origin, the other is the distance $y \geq 0$ of m_3 to the origin, and the third is the distance $z \geq 0$ of m_5 to the origin (the distances of m_2, m_4 and m_6 to the origin are obtained by symmetry). The system has seven singularities, the triple collisions between m_0, m_1 and m_2, between m_0, m_3 and m_4, and between m_0, m_5 and m_6;

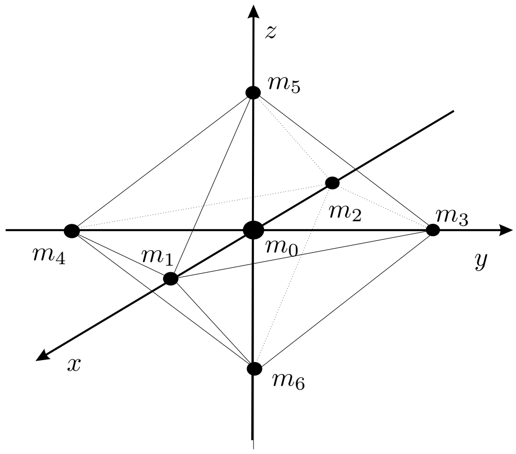

FIGURE 1. The octahedral 7-body problem.

the quintuple collisions between m_0, m_1, m_2, m_3 and m_4, between m_0, m_1, m_2, m_5 and m_6, and between m_0, m_3, m_4, m_5 and m_6; and finally the total collision of the seven bodies. Due to the symmetries doing a triple Levi–Civita transformation we regularize the three triple collisions.

When $\mu = 0$ using the symmetry the problem is reduced to three collision 2-body problems, between m_0 and m_1, between m_0 and m_3, and between m_0 and m_5. Note that if we take into account the seven bodies, then really for $\mu = 0$ we have instead of the binary collisions m_0 with m_1, m_0 with m_3 and m_0 with m_5, the triple collisions m_0, m_1 and m_2, m_0, m_3 and m_4, and m_0, m_5 and m_6 respectively. Since the solutions of the collision 2-body problem are known we can compute the periodic solutions of the regularized system for $\mu = 0$ in a fixed negative energy level $h < 0$. The objective of this paper is to prove that the symmetric periodic orbits of the regularized octahedral 7-body problem for $\mu = 0$ can be continued to symmetric periodic orbits of the regularized octahedral 7-body problem for $\mu > 0$ sufficiently small. The main tool for proving this result is the classical analytic continuation method of Poincaré.

In [5] applying the techniques applied in this paper the authors prove the existence of infinitely many symmetric periodic orbits of the the planar regularized rhomboidal 5-body problem for $\mu > 0$ sufficiently small. The regularized octahedral 7-body problem has more symmetries and consequently is richer in families of symmetric periodic orbits. The techniques applied here are also similar to the ones applied in [4] and [6] in order to prove the existence of infinitely many periodic orbits for the collinear 3-body problem.

The paper is structured as follows. In Section 2 we give the equations of motion of the octahedral 7-body problem and we apply a triple Levi–Civita transformation to regularize the triple collisions between m_0, m_1 and m_2, between m_0,

m_3 and m_4 and between m_0, m_5 and m_6. Notice that the quintuple collisions and the total collision are not regularized.

In Section 3 we analyze the discrete symmetries of the regularized problem. In particular, we see that there are four different symmetries that provide symmetric periodic orbits of the problem. In fact all the symmetric periodic orbits that we find are *double symmetric periodic orbits*; i.e., periodic orbits which are simultaneously symmetric with respect to exactly two different symmetries. Moreover we see that there are no *triple* and no *quadruple symmetric periodic orbits*; i.e., periodic orbits which are simultaneously symmetric with respect to three or four symmetries.

In Section 4 we compute the periodic orbits of the regularized octahedral 7-body problem for $\mu = 0$ and we analyze its symmetric periodic orbits. In particular we will see that all symmetric periodic orbits of the regularized octahedral 7-body problem for $\mu = 0$ are double symmetric periodic orbits.

Finally in Section 5 we apply the analytic continuation method of Poincaré to continue the double symmetric periodic orbits of the regularized octahedral 7-body problem for $\mu = 0$ to double–symmetric periodic orbits of the regularized octahedral 7-body problem for $\mu > 0$ sufficiently small.

2. Equations of motion for the octahedral 7-body problem

We consider seven point particles with masses $m_0 = 1$, $m_1 = m_2 = \mu\,\nu_1$, $m_3 = m_4 = \mu\,\nu_2$, $m_5 = m_6 = \mu\,\nu_3$ positions $\mathbf{q}_0 = (0,0,0)$, $\mathbf{q}_1 = (x,0,0)$, $\mathbf{q}_2 = (-x,0,0)$, $\mathbf{q}_3 = (0,y,0)$, $\mathbf{q}_4 = (0,-y,0)$, $\mathbf{q}_5 = (0,0,z)$, and $\mathbf{q}_6 = (0,0,-z)$, respectively, and velocities $\mathbf{v}_0 = (0,0,0)$, $\mathbf{v}_1 = (v_x,0,0)$, $\mathbf{v}_2 = (-v_x,0,0)$, $\mathbf{v}_3 = (0,v_y,0)$, $\mathbf{v}_4 = (0,-v_y,0)$, $\mathbf{v}_5 = (0,0,v_z)$, and $\mathbf{v}_6 = (0,0,-v_z)$ respectively (see Figure 1). Here $x, y, z \in [0,+\infty)$ and $v_x, v_y, v_z \in \mathbb{R}$. Our 7-body problem consists of describing the motion of these particles under their mutual Newtonian gravitational attractions. We note that due to the symmetry of the problem the mass m_0 rest at the origin at any time and the motion of the masses m_1 and m_2 is confined to the x-axis, the motion of the masses m_3 and m_4 is confined to the y-axis, and the motion of the masses m_5 and m_6 is confined to the z-axis. Since the configuration of the six bodies in motion is always an octahedron with center at m_0, we call the study of the motion of this 7-body problem the *octahedral* 7*-body problem.*

Without loss of generality we can assume that the gravitational constant is $G = 1$. Then the kinetic energy of the octahedral 7-body problem is

$$T = \mu\,\nu_1\,\dot{x}^2 + \mu\,\nu_2\,\dot{y}^2 + \mu\,\nu_3\,\dot{z}^2\,,$$

where the dot denotes derivative with respect to the time t and the potential energy is

$$\begin{aligned} U = &-\frac{\mu\,\nu_1\,(4+\mu\,\nu_1)}{2x} - \frac{\mu\,\nu_2\,(4+\mu\,\nu_2)}{2y} \\ &- \frac{\mu\,\nu_3\,(4+\mu\,\nu_3)}{2z} - \frac{4\mu^2\,\nu_1\,\nu_2}{\sqrt{x^2+y^2}} - \frac{4\mu^2\,\nu_1\,\nu_3}{\sqrt{x^2+z^2}} - \frac{4\mu^2\,\nu_2\,\nu_3}{\sqrt{y^2+z^2}}\,. \end{aligned}$$

The Lagrangian of the problem is given by $L = T - U$. By the Legendre transformation (see for instance [1, 2, 7]) the Hamiltonian of the problem is

$$H = \frac{p_x^2}{4\mu\,\nu_1} + \frac{p_y^2}{4\mu\,\nu_2} + \frac{p_z^2}{4\mu\,\nu_3} + U\,,$$

where p_x, p_y and p_z are the conjugate momenta. The equations of motion associated to the Hamiltonian H are

$$\begin{aligned}
\dot{x} &= \frac{p_x}{2\mu\nu_1}\,, \quad \dot{p_x} = -\frac{\mu\,\nu_1\,(4+\mu\,\nu_1)}{2x^2} - \frac{4\mu^2\,\nu_1\,\nu_2\,x}{(x^2+y^2)^{3/2}} - \frac{4\mu^2\,\nu_1\,\nu_3\,x}{(x^2+z^2)^{3/2}}\,,\\
\dot{y} &= \frac{p_y}{2\mu\nu_2}\,, \quad \dot{p_y} = -\frac{\mu\,\nu_2\,(4+\mu\,\nu_2)}{2y^2} - \frac{4\mu^2\,\nu_1\,\nu_2\,y}{(x^2+y^2)^{3/2}} - \frac{4\mu^2\,\nu_2\,\nu_3\,y}{(y^2+z^2)^{3/2}}\,,\\
\dot{z} &= \frac{p_z}{2\mu\nu_3}\,, \quad \dot{p_z} = -\frac{\mu\,\nu_3\,(4+\mu\,\nu_3)}{2z^2} - \frac{4\mu^2\,\nu_1\,\nu_3\,z}{(x^2+z^2)^{3/2}} - \frac{4\mu^2\,\nu_2\,\nu_3\,z}{(y^2+z^2)^{3/2}}\,.
\end{aligned} \tag{1}$$

Doing the rescaling of the variables $x = \mu^2 X$, $y = \mu^2 Y$, $z = \mu^2 Z$ and $t = \mu^3 T$, and denoting the new variables (X, Y, Z, T) again by (x, y, z, t) system (1) becomes

$$\begin{aligned}
\dot{x} &= \frac{p_x}{2\nu_1}\,, \quad \dot{p_x} = -\frac{\nu_1\,(4+\mu\,\nu_1)}{2x^2} - \frac{4\mu\,\nu_1\,\nu_2\,x}{(x^2+y^2)^{3/2}} - \frac{4\mu\,\nu_1\,\nu_3\,x}{(x^2+z^2)^{3/2}}\,,\\
\dot{y} &= \frac{p_y}{2\nu_2}\,, \quad \dot{p_y} = -\frac{\nu_2\,(4+\mu\,\nu_2)}{2y^2} - \frac{4\mu\,\nu_1\,\nu_2\,y}{(x^2+y^2)^{3/2}} - \frac{4\mu\,\nu_2\,\nu_3\,y}{(y^2+z^2)^{3/2}}\,,\\
\dot{z} &= \frac{p_z}{2\nu_3}\,, \quad \dot{p_z} = -\frac{\nu_3\,(4+\mu\,\nu_3)}{2z^2} - \frac{4\mu\,\nu_1\,\nu_3\,z}{(x^2+z^2)^{3/2}} - \frac{4\mu\,\nu_2\,\nu_3\,z}{(y^2+z^2)^{3/2}}\,.
\end{aligned} \tag{2}$$

This system is also Hamiltonian with Hamiltonian

$$\begin{aligned}
H = {} & \frac{p_x^2}{4\nu_1} + \frac{p_y^2}{4\nu_2} + \frac{p_z^2}{4\nu_3} - \frac{\nu_1\,(4+\mu\,\nu_1)}{2x} - \frac{\nu_2\,(4+\mu\,\nu_2)}{2y} - \frac{\nu_3\,(4+\mu\,\nu_3)}{2z}\\
& - \frac{4\mu\,\nu_1\,\nu_2}{\sqrt{x^2+y^2}} - \frac{4\mu\,\nu_1\,\nu_3}{\sqrt{x^2+z^2}} - \frac{4\mu\,\nu_2\,\nu_3}{\sqrt{y^2+z^2}}\,,
\end{aligned}$$

We note that system (2) has seven singularities: $x = 0$, that corresponds to tripe collision between m_0, m_1 and m_2, $y = 0$ that corresponds to triple collision between m_0, m_3 and m_4, $z = 0$ that corresponds to triple collision between m_0, m_5 and m_6, $x^2 + y^2 = 0$ that corresponds to the quintuple collision between m_0, m_1, m_2, m_3 and m_4, $x^2 + z^2 = 0$ that corresponds to the quintuple collision between m_0, m_1, m_2, m_5 and m_6, $y^2 + z^2 = 0$ that corresponds to the quintuple collision between m_0, m_3, m_4, m_5 and m_6, and finally $x^2 + y^2 + z^2 = 0$ that corresponds to the total collision of the seven masses. We regularize the three triple collisions applying a triple Levi–Civita transformation (see [3, 6, 9])

$$x = \xi_1^2\,, \quad y = \xi_2^2\,, \quad z = \xi_3^2\,, \quad p_x = \frac{\eta_1}{2\xi_1}\,, \quad p_y = \frac{\eta_2}{2\xi_2}\,, \quad p_z = \frac{\eta_3}{2\xi_3}\,, \quad dt = 4\xi_1^2\xi_2^2\xi_3^2\,ds\,.$$

The regularized system of the octahedral 7-body problem (2) on the level energy $H = h$ for some constant h is the Hamiltonian system

$$
\begin{aligned}
\frac{d\xi_1}{ds} &= \frac{\eta_1\xi_2^2\xi_3^2}{2\nu_1}\,,\\
\frac{d\xi_2}{ds} &= \frac{\eta_2\xi_1^2\xi_3^2}{2\nu_2}\,,\\
\frac{d\xi_3}{ds} &= \frac{\eta_3\xi_1^2\xi_2^2}{2\nu_3}\,,\\
\frac{d\eta_1}{ds} &= -\frac{\xi_1\xi_3^2\eta_2^2}{2\nu_2} - \frac{\xi_1\xi_2^2\eta_3^2}{2\nu_3} + 4\nu_2(4+\mu\nu_2)\xi_1\xi_3^2 + 4\nu_3(4+\mu\nu_3)\xi_1\xi_2^2\\
&\quad + 8h\xi_1\xi_2^2\xi_3^2 + \frac{32\mu\nu_1\nu_2\xi_1\xi_2^6\xi_3^2}{(\xi_1^4+\xi_2^4)^{3/2}} + \frac{32\mu\nu_1\nu_3\xi_1\xi_2^2\xi_3^6}{(\xi_1^4+\xi_3^4)^{3/2}} + \frac{32\mu\nu_2\nu_3\xi_1\xi_2^2\xi_3^2}{(\xi_2^4+\xi_3^4)^{1/2}}\,, \qquad (3)\\
\frac{d\eta_2}{ds} &= -\frac{\xi_2\xi_3^2\eta_1^2}{2\nu_1} - \frac{\xi_1^2\xi_2\eta_3^2}{2\nu_3} + 4\nu_1(4+\mu\nu_1)\xi_2\xi_3^2 + 4\nu_3(4+\mu\nu_3)\xi_1^2\xi_2\\
&\quad + 8h\xi_1^2\xi_2\xi_3^2 + \frac{32\mu\nu_1\nu_2\xi_1^6\xi_2\xi_3^2}{(\xi_1^4+\xi_2^4)^{3/2}} + \frac{32\mu\nu_2\nu_3\xi_1^2\xi_2\xi_3^6}{(\xi_2^4+\xi_3^4)^{3/2}} + \frac{32\mu\nu_1\nu_3\xi_1^2\xi_2\xi_3^2}{(\xi_1^4+\xi_3^4)^{1/2}}\,,\\
\frac{d\eta_3}{ds} &= -\frac{\xi_2^2\xi_3\eta_1^2}{2\nu_1} - \frac{\xi_1^2\xi_3\eta_2^2}{2\nu_2} + 4\nu_1(4+\mu\nu_1)\xi_2^2\xi_3 + 4\nu_2(4+\mu\nu_2)\xi_1^2\xi_3\\
&\quad + 8h\xi_1^2\xi_2^2\xi_3 + \frac{32\mu\nu_1\nu_3\xi_1^6\xi_2^2\xi_3}{(\xi_1^4+\xi_3^4)^{3/2}} + \frac{32\mu\nu_2\nu_3\xi_1^2\xi_2^6\xi_3}{(\xi_2^4+\xi_3^4)^{3/2}} + \frac{32\mu\nu_1\nu_2\xi_1^2\xi_2^2\xi_3}{(\xi_1^4+\xi_2^4)^{1/2}}\,.
\end{aligned}
$$

with Hamiltonian

$$
\begin{aligned}
K = {} & \frac{\eta_1^2\xi_2^2\xi_3^2}{4\nu_1} + \frac{\eta_2^2\xi_1^2\xi_3^2}{4\nu_2} + \frac{\eta_3^2\xi_1^2\xi_2^2}{4\nu_3} - 2\nu_1(4+\mu\nu_1)\xi_2^2\xi_3^2 - 2\nu_2(4+\mu\nu_2)\xi_1^2\xi_3^2\\
& - 2\nu_3(4+\mu\nu_3)\xi_1^2\xi_2^2 - 4h\xi_1^2\xi_2^2\xi_3^2 - \frac{16\mu\nu_1\nu_2\xi_1^2\xi_2^2\xi_3^2}{\sqrt{\xi_1^4+\xi_2^4}} - \frac{16\mu\nu_1\nu_3\xi_1^2\xi_2^2\xi_3^2}{\sqrt{\xi_1^4+\xi_3^4}}\\
& - \frac{16\mu\nu_2\nu_3\xi_1^2\xi_2^2\xi_3^2}{\sqrt{\xi_2^4+\xi_3^4}}\,,
\end{aligned}
$$

and satisfies the energy relation $K = 0$; i.e., $H = h$.

We note that system (3) is analytic with respect to its variables except when $\xi_1^4+\xi_2^4 = 0$, $\xi_1^4+\xi_3^4 = 0$, $\xi_2^4+\xi_3^4 = 0$ and $\xi_1^2+\xi_2^2+\xi_3^2 = 0$, which correspond to the three quintuple collisions and to the total collision, respectively.

The regularization of the triple collisions allows us to look for periodic orbits of the octahedral 7-body problem containing triple collisions. Our aim is to find periodic orbits of the octahedral 7-body problem (3) for $\mu > 0$ sufficiently small, satisfying the energy relation $K = 0$. In fact, we look only for symmetric periodic orbits which are easier to study than the general periodic orbits.

3. Symmetries

It is easy to check that system (3) is invariant under the discrete symmetries

$$
\begin{aligned}
S_1 &: (\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3, s) \longrightarrow (\xi_1, \xi_2, \xi_3, -\eta_1, -\eta_2, -\eta_3, -s)\,,\\
S_2 &: (\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3, s) \longrightarrow (-\xi_1, \xi_2, \xi_3, \eta_1, -\eta_2, -\eta_3, -s)\,,\\
S_3 &: (\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3, s) \longrightarrow (\xi_1, -\xi_2, \xi_3, -\eta_1, \eta_2, -\eta_3, -s)\,,\\
S_4 &: (\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3, s) \longrightarrow (\xi_1, \xi_2, -\xi_3, -\eta_1, -\eta_2, \eta_3, -s)\,,\\
S_5 &: (\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3, s) \longrightarrow (\xi_1, -\xi_2, -\xi_3, -\eta_1, \eta_2, \eta_3, -s)\,,\\
S_6 &: (\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3, s) \longrightarrow (-\xi_1, \xi_2, -\xi_3, \eta_1, -\eta_2, \eta_3, -s)\,,\\
S_7 &: (\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3, s) \longrightarrow (-\xi_1, -\xi_2, \xi_3, \eta_1, \eta_2, -\eta_3, -s)\,,\\
S_8 &: (\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3, s) \longrightarrow (-\xi_1, -\xi_2, -\xi_3, \eta_1, \eta_2, \eta_3, -s)\,.
\end{aligned}
$$

The invariance under these symmetries means that if $\varphi(s) = (\xi_1(s), \xi_2(s), \xi_3(s), \eta_1(s), \eta_2(s), \eta_3(s))$ is a solution of system (3), then also $S_i(\varphi(s))$ is a solution for $i = 1, \ldots, 8$. An orbit $\varphi(s)$ is called *S_i-symmetric* if $S_i(\varphi(s)) = \varphi(s)$.

Using the uniqueness theorem of a solution of an ordinary differential system, if $\varphi(s)$ is such that $\eta_1(0) = \eta_2(0) = \eta_3(0) = 0$, then $\varphi(s)$ is a S_1-symmetric solution. If in addition there exists $\overline{s} > 0$ such that $\eta_1(\overline{s}) = \eta_2(\overline{s}) = \eta_3(\overline{s}) = 0$ and $\eta_1(s)$, $\eta_2(s)$ and $\eta_3(s)$ are not simultaneously zero for all $s \in (0, \overline{s})$, then $\varphi(s)$ is a S_1-symmetric periodic solution of period $2\overline{s}$. Using similar arguments for the other symmetries, we obtain the following proposition. Notice that system (3) is autonomous, therefore the origin of time can be chosen arbitrarily.

Proposition 1. *Let $\varphi(s) = (\xi_1(s), \xi_2(s), \xi_3(s), \eta_1(s), \eta_2(s), \eta_3(s))$ be a solution of* (3).

(a) *If $\eta_1(s)$, $\eta_2(s)$ and $\eta_3(s)$ are zero at $s = s_0$ and at $s = s_0 + S/2$ but they are not simultaneously zero at any value of $s \in (s_0, s_0 + S/2)$, then $\varphi(s)$ is a S_1-symmetric periodic solution of period S.*

(b) *If $\xi_1(s)$, $\eta_2(s)$ and $\eta_3(s)$ are zero at $s = s_0$ and at $s = s_0 + S/2$ but they are not simultaneously zero at any value of $s \in (s_0, s_0 + S/2)$, then $\varphi(s)$ is a S_2-symmetric periodic solution of period S.*

(c) *If $\xi_2(s)$, $\eta_1(s)$ and $\eta_3(s)$ are zero at $s = s_0$ and at $s = s_0 + S/2$ but they are not simultaneously zero at any value of $s \in (s_0, s_0 + S/2)$, then $\varphi(s)$ is a S_3-symmetric periodic solution of period S.*

(d) *If $\xi_3(s)$, $\eta_1(s)$ and $\eta_2(s)$ are zero at $s = s_0$ and at $s = s_0 + S/2$ but they are not simultaneously zero at any value of $s \in (s_0, s_0 + S/2)$, then $\varphi(s)$ is a S_4-symmetric periodic solution of period S.*

Since in system (3) the quintuple collisions are not regularized, in our study we must avoid the orbits of the octahedral 7-body problem which start or end at quintuple collision; that is, we must avoid orbits such that either $\xi_1(s) = \xi_2(s) = 0$, or $\xi_1(s) = \xi_3(s) = 0$, or $\xi_2(s) = \xi_3(s) = 0$ for some s. For this reason the symmetries S_5, S_6, S_7 and S_8 are not considered.

There could be periodic solutions of (3) that are symmetric exactly with respect to two symmetries. These kinds of symmetric periodic solutions are characterized in the following result.

Proposition 2. *Let* $\varphi(s) = (\xi_1(s), \xi_2(s), \xi_3(s), \eta_1(s), \eta_2(s), \eta_3(s))$ *be a solution of the octahedral* 7*-body problem* (3).

(a) *The solution* $\varphi(s)$ *is a* S_{12}*-symmetric periodic solution of period* S *if and only if* $\eta_1(s_0) = \eta_2(s_0) = \eta_3(s_0) = 0$ *and* $\xi_1(s_0 + S/4) = \eta_2(s_0 + S/4) = \eta_3(s_0 + S/4) = 0$*, and there is no* $s \in (s_0, s_0 + S/4)$ *satisfying that* $\xi_1(s) = \eta_2(s) = \eta_3(s) = 0$.

(b) *The solution* $\varphi(s)$ *is a* S_{13}*-symmetric periodic solution of period* S *if and only if* $\eta_1(s_0) = \eta_2(s_0) = \eta_3(s_0) = 0$ *and* $\xi_2(s_0 + S/4) = \eta_1(s_0 + S/4) = \eta_3(s_0 + S/4) = 0$*, and there is no* $s \in (s_0, s_0 + S/4)$ *satisfying that* $\xi_2(s) = \eta_1(s) = \eta_3(s) = 0$.

(c) *The solution* $\varphi(s)$ *is a* S_{14}*-symmetric periodic solution of period* S *if and only if* $\eta_1(s_0) = \eta_2(s_0) = \eta_3(s_0) = 0$ *and* $\xi_3(s_0 + S/4) = \eta_1(s_0 + S/4) = \eta_2(s_0 + S/4) = 0$*, and there is no* $s \in (s_0, s_0 + S/4)$ *satisfying that* $\xi_3(s) = \eta_1(s) = \eta_2(s) = 0$.

(d) *The solution* $\varphi(s)$ *is a* S_{23}*-symmetric periodic solution of period* S *if and only if* $\xi_1(s_0) = \eta_2(s_0) = \eta_3(s_0) = 0$ *and* $\xi_2(s_0 + S/4) = \eta_1(s_0 + S/4) = \eta_3(s_0 + S/4) = 0$*, and there is no* $s \in (s_0, s_0 + S/4)$ *satisfying that* $\xi_2(s) = \eta_1(s) = \eta_3(s) = 0$.

(e) *The solution* $\varphi(s)$ *is a* S_{24}*-symmetric periodic solution of period* S *if and only if* $\xi_1(s_0) = \eta_2(s_0) = \eta_3(s_0) = 0$ *and* $\xi_3(s_0 + S/4) = \eta_1(s_0 + S/4) = \eta_2(s_0 + S/4) = 0$*, and there is no* $s \in (s_0, s_0 + S/4)$ *satisfying that* $\xi_3(s) = \eta_1(s) = \eta_2(s) = 0$.

(f) *The solution* $\varphi(s)$ *is a* S_{34}*-symmetric periodic solution of period* S *if and only if* $\xi_2(s_0) = \eta_1(s_0) = \eta_3(s_0) = 0$ *and* $\xi_3(s_0 + S/4) = \eta_1(s_0 + S/4) = \eta_2(s_0 + S/4) = 0$*, and there is no* $s \in (s_0, s_0 + S/4)$ *satisfying that* $\xi_3(s) = \eta_1(s) = \eta_2(s) = 0$.

We note that in Proposition 2 we only give the conditions for the S_{ij}-symmetric periodic solutions which satisfy the conditions of symmetry S_i at time s_0 and the conditions of symmetry S_j at time $s_0 + S/4$. Obviously if a solution satisfies the conditions of symmetry S_j at time s_0 and the conditions of symmetry S_i at time $s_0 + S/4$ it is also a S_{ij}-symmetric periodic solution.

The next result shows that there are no symmetric periodic orbits that are symmetric with respect to three or four symmetries.

Proposition 3. *There are no periodic solutions of the octahedral* 7*-body problem* (3)*, which are simultaneously symmetric by three or four symmetries.*

Proof. Assume that $\varphi(s)$ is a S_i-symmetric periodic solution of period S for $i = 1, 2, 3$. Then there exist times s_1, s_2 and s_3 with $s_1, s_2, s_3 \in [0, S/2)$ such that:

$$\eta_1(s_1) = \eta_2(s_1) = \eta_3(s_1) = 0\,, \qquad \xi_1(s_2) = \eta_2(s_2) = \eta_3(s_2) = 0\,,$$
$$\xi_2(s_3) = \eta_1(s_2) = \eta_3(s_3) = 0\,.$$

We assume that $s_1 = 0$. This is not restrictive because system (3) is autonomous, and consequently the origin of time can be chosen arbitrarily. Since the orbit is in particular S_{12}-symmetric, from Proposition 2, $s_2 = S/4$. Similarly, since it is also S_{13}-symmetric, again from Proposition 2, $s_3 = S/4$. The fact that $s_2 = s_3$ is a contradiction because $\xi_1(s_2) = \xi_2(s_2) = 0$ implies the existence of a quintuple collision but, since this kind of collisions are not regularized, there are no periodic orbits with such collisions. So we have proved that there are no S_{123}-symmetric periodic solutions. The other cases can be proved in a similar way. $\square$

4. Symmetric periodic solutions for $\mu = 0$

For $\mu = 0$ system (3) becomes

$$\begin{aligned}
\frac{d\xi_1}{ds} &= \frac{\eta_1\xi_2^2\xi_3^2}{2\nu_1}\,,\\
\frac{d\xi_2}{ds} &= \frac{\eta_2\xi_1^2\xi_3^2}{2\nu_2}\,,\\
\frac{d\xi_3}{ds} &= \frac{\eta_3\xi_1^2\xi_2^2}{2\nu_3}\,,\\
\frac{d\eta_1}{ds} &= -\frac{\xi_1\xi_3^2\eta_2^2}{2\nu_2} - \frac{\xi_1\xi_2^2\eta_3^2}{2\nu_3} + 16\nu_2\xi_1\xi_3^2 + 16\nu_3\xi_1\xi_2^2 + 8h\xi_1\xi_2^2\xi_3^2\,,\\
\frac{d\eta_2}{ds} &= -\frac{\xi_2\xi_3^2\eta_1^2}{2\nu_1} - \frac{\xi_1^2\xi_2\eta_3^2}{2\nu_3} + 16\nu_1\xi_2\xi_3^2 + 16\nu_3\xi_1^2\xi_2 + 8h\xi_1^2\xi_2\xi_3^2\,,\\
\frac{d\eta_3}{ds} &= -\frac{\xi_2^2\xi_3\eta_1^2}{2\nu_1} - \frac{\xi_1^2\xi_3\eta_2^2}{2\nu_2} + 16\nu_1\xi_2^2\xi_3 + 16\nu_2\xi_1^2\xi_3 + 8h\xi_1^2\xi_2^2\xi_3\,,
\end{aligned} \tag{4}$$

and the Hamiltonian K goes over to

$$K = \frac{\eta_1^2\xi_2^2\xi_3^2}{4\nu_1} + \frac{\eta_2^2\xi_1^2\xi_3^2}{4\nu_2} + \frac{\eta_3^2\xi_1^2\xi_2^2}{4\nu_3} - 8\nu_1\xi_2^2\xi_3^2 - 8\nu_2\xi_1^2\xi_3^2 - 8\nu_3\xi_1^2\xi_2^2 - 4h\xi_1^2\xi_2^2\xi_3^2\,.$$

The Hamiltonian H for $\mu = 0$ can be written as

$$\begin{aligned}
H &= H_1(x,p_x) + H_2(y,p_y) + H_3(z,p_z)\\
&= \left(\frac{p_x^2}{4\nu_1} - \frac{2\nu_1}{x}\right) + \left(\frac{p_y^2}{4\nu_2} - \frac{2\nu_2}{y}\right) + \left(\frac{p_z^2}{4\nu_3} - \frac{2\nu_3}{z}\right)\,.
\end{aligned}$$

We note that $H_1(x,p_x)$, $H_2(y,p_y)$ and $H_3(z,p_z)$ are three fist integrals of the nonregularized problem, so they are constant along the solutions in the intervals between two consecutive zeros of x, y and z respectively.

The flow of the octahedral 7-body problem on the energy level $H = h$ for some constant h is obtained from the flow of the Hamiltonian $H_1(x,p_x)$ on the energy level $H_1 = h_1$, from the flow of the Hamiltonian $H_2(y,p_y)$ on the energy level $H_2 = h_2$, and from the flow of the Hamiltonian $H_3(z,p_z)$ on the energy level $H_3 = h_3$ with $h = h_1 + h_2 + h_3$.

The Hamiltonian $H_i(x, p_x)$ in the Levi–Civita coordinates (ξ_i, η_i) is given by

$$H_i = \frac{\eta_i^2}{16\nu_i\xi_i^2} - \frac{2\nu_i}{\xi_i^2} = h_i \,,$$

for $i = 1, 2, 3$.

Let $(\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3)$ be a solution of (4) satisfying the energy relation $K = 0$ (i.e., $H = h = h_1 + h_2 + h_3$ with $H_1 = h_1$, $H_2 = h_2$ and $H_3 = h_3$), we define a new time variable σ as follows

$$\frac{d\sigma}{ds} = \xi_2^2\xi_3^2 \,, \quad \text{or equivalently} \quad \frac{dt}{d\sigma} = 4\xi_1^2 \,. \tag{5}$$

The Hamiltonian K in the new time variable σ can be written as

$$\begin{aligned} K_1 &= \frac{1}{\xi_2^2\xi_3^2}\,K \\ &= \frac{\eta_1^2}{4\nu_1} - 8\nu_1 - 4h_1\xi_1^2 + \xi_1^2\left(\frac{\eta_2^2}{4\nu_2\,\xi_2^2} - \frac{8\nu_2}{\xi_2^2} - 4h_2\right) + \xi_1^2\left(\frac{\eta_3^2}{4\nu_3\,\xi_3^2} - \frac{8\nu_3}{\xi_3^2} - 4h_3\right) \\ &= \frac{\eta_1^2}{4\nu_1} - 8\nu_1 - 4h_1\xi_1^2 \,. \end{aligned}$$

Then ξ_1, η_1 satisfy the system of differential equations associated to the Hamiltonian K_1

$$\frac{d\xi_1}{d\sigma} = \frac{\eta_1}{2\nu_1} \,, \quad \frac{d\eta_1}{d\sigma} = 8\,h_1\,\xi_1 \,, \tag{6}$$

and the energy relation $K_1 = 0$.

We are only interested in periodic solutions of (6). Thus we must consider only negative values of h_1. Then, fixed $h_1 < 0$, system (6) can be integrated directly and the solution $(\xi_1(\sigma), \eta_1(\sigma))$ of (6) with initial conditions

$$\xi_1(0) = \xi_{10}^* \,, \quad \eta_1(0) = \eta_{10}^* \,, \tag{7}$$

is

$$\begin{aligned} \xi_1(\sigma) &= \xi_{10}^*\,\cos(w_1\,\sigma) + \frac{\eta_{10}^*}{2w_1\nu_1}\,\sin(w_1\,\sigma) \,, \\ \eta_1(\sigma) &= \eta_{10}^*\,\cos(w_1\,\sigma) - 2w_1\nu_1\xi_{10}^*\,\sin(w_1\,\sigma) \,, \end{aligned} \tag{8}$$

where $w_1 = 2\sqrt{-h_1/\nu_1}$.

We note that (8) is a periodic solution of (6) with period $\overline{\sigma} = 2\pi/w_1$. Since we are interested in the periodic solution (8) satisfying the energy relation $K_1 = 0$, by (5), the period of (8) in the real time t is given by

$$T_1(h_1, \nu_1) = \int_0^{\overline{\sigma}} 4\,\xi_1^2(\sigma)\,d\sigma = 4\,\pi\left(-\frac{\nu_1}{h_1}\right)^{3/2} .$$

Now we introduce a new time τ with

$$\frac{d\tau}{ds} = \xi_1^2\xi_3^2 \,, \quad \text{or equivalently} \quad \frac{dt}{d\tau} = 4\xi_2^2 \,. \tag{9}$$

Then ξ_2, η_2 in the new time τ are solutions of the Hamiltonian system

$$\frac{d\xi_2}{d\tau} = \frac{\eta_2}{2\nu_2}\,, \qquad \frac{d\eta_2}{d\tau} = 8\,h_2\,\xi_2\,, \tag{10}$$

with Hamiltonian

$$K_2 = \frac{1}{\xi_1^2\xi_3^2}\,K = \frac{\eta_2^2}{4\,\nu_2} - 8\,\nu_2 - 4\,h_2\,\xi_2^2\,,$$

that satisfies de energy relation $K_2 = 0$. Moreover, fixed $h_2 < 0$, the solution $(\xi_2(\tau), \eta_2(\tau))$ of (10) with initial conditions

$$\xi_2(0) = \xi_{20}^*\,, \quad \eta_2(0) = \eta_{20}^*\,, \tag{11}$$

is given by

$$\begin{aligned} \xi_2(\tau) &= \xi_{20}^*\,\cos(w_2\,\tau) + \frac{\eta_{20}^*}{2w_2\nu_2}\,\sin(w_2\,\tau)\,, \\ \eta_2(\tau) &= \eta_{20}^*\,\cos(w_2\,\tau) - 2w_2\nu_2\xi_{20}^*\,\sin(w_2\,\tau)\,, \end{aligned} \tag{12}$$

where $w_2 = 2\sqrt{-h_2/\nu_2}$.

The solution (12) is periodic of period $\overline{\tau} = 2\pi/w_2$. Moreover, since (12) satisfies the energy relation $K_2 = 0$, by (9), the period of (12) in the real time t is given by

$$T_2(h_2, \nu_2) = \int_0^{\overline{\tau}} 4\,\xi_2^2(\tau)\,d\tau = 4\,\pi\left(-\frac{\nu_2}{h_2}\right)^{3/2}.$$

Finally we introduce a new time υ with

$$\frac{d\upsilon}{ds} = \xi_1^2\xi_2^2\,, \quad \text{or equivalently} \quad \frac{dt}{d\upsilon} = 4\xi_3^2\,. \tag{13}$$

Then ξ_3, η_3 in the new time υ are solutions of the Hamiltonian system

$$\frac{d\xi_3}{d\upsilon} = \frac{\eta_3}{2\nu_3}\,, \qquad \frac{d\eta_3}{d\upsilon} = 8\,h_3\,\xi_3\,, \tag{14}$$

with Hamiltonian

$$K_3 = \frac{1}{\xi_1^2\xi_2^2}\,K = \frac{\eta_3^2}{4\,\nu_3} - 8\,\nu_3 - 4\,h_3\,\xi_3^2\,,$$

that satisfies de energy relation $K_3 = 0$. Moreover, fixed $h_3 < 0$, the solution $(\xi_3(\upsilon), \eta_3(\upsilon))$ of (14) with initial conditions

$$\xi_3(0) = \xi_{30}^*\,, \quad \eta_3(0) = \eta_{30}^*\,, \tag{15}$$

is given by

$$\begin{aligned} \xi_3(\upsilon) &= \xi_{30}^*\,\cos(w_3\,\upsilon) + \frac{\eta_{30}^*}{2w_3\nu_3}\,\sin(w_3\,\upsilon)\,, \\ \eta_3(\upsilon) &= \eta_{30}^*\,\cos(w_3\,\upsilon) - 2w_3\nu_3\xi_{30}^*\,\sin(w_3\,\upsilon)\,, \end{aligned} \tag{16}$$

where $w_3 = 2\sqrt{-h_3/\nu_3}$.

TABLE 1. Period of $\varphi(s)$.

Time t	Time σ	Time τ	Time υ	Time s
$T = p\,T_1(h_1,\nu_1)$ $= q\,T_2(h_2,\nu_2)$ $= \ell\,T_3(h_3,\nu_3)$	$\sigma^* = p\,\overline{\sigma}$	$\tau^* = q\,\overline{\tau}$	$\upsilon^* = \ell\,\overline{\upsilon}$	$S^* = s(T)$
$T/4$	$\sigma^*/4$	$\tau^*/4$	$\upsilon^*/4$	$S^*/4$

The solution (16) is periodic of period $\overline{\upsilon} = 2\pi/w_3$. Moreover, since (16) satisfies the energy relation $K_3 = 0$, by (13), the period of (16) in the real time t is given by

$$T_3(h_3,\nu_3) = \int_0^{\overline{\upsilon}} 4\,\xi_3^2(\upsilon)\,d\upsilon = 4\,\pi\left(-\frac{\nu_3}{h_3}\right)^{3/2}.$$

Proposition 4. *Let $(\xi_1(\sigma),\eta_1(\sigma))$ be a periodic solution of* (6)*, for a fixed $h_1 < 0$, with initial conditions* (7) *and period $\overline{\sigma} = 2\pi/w_1$ that satisfies $K_1 = 0$. Let $(\xi_2(\tau),\eta_2(\tau))$ be the periodic solution of* (10)*, for a fixed $h_2 < 0$, with initial conditions* (11) *and period $\overline{\tau} = 2\pi/w_2$ that satisfies $K_2 = 0$. Let $(\xi_3(\upsilon),\eta_3(\upsilon))$ be the periodic solution of* (14)*, for a fixed $h_3 < 0$, with initial conditions* (15) *and period $\overline{\upsilon} = 2\pi/w_3$ that satisfies $K_3 = 0$. Assume that $h = h_1 + h_2 + h_3$ and that $\sigma(s)$, $\tau(s)$, $\upsilon(s)$ are given by* (5), (9) *and* (13) *respectively, where we choose $\sigma(0) = \tau(0) = \upsilon(0) = 0$. Suppose that there is no $s \in \mathbb{R}$ such that $\xi_1(\sigma(s)) = \xi_2(\tau(s)) = 0$, $\xi_1(\sigma(s)) = \xi_3(\upsilon(s)) = 0$ and $\xi_2(\tau(s)) = \xi_3(\upsilon(s)) = 0$. Then the following statements hold.*

(a) *$\varphi(s) = (\xi_1(\sigma(s)),\xi_2(\tau(s)),\xi_3(\upsilon(s)),\eta_1(\sigma(s)),\eta_2(\tau(s)),\eta_3(\upsilon(s)))$ is a solution of* (4) *with initial conditions $\xi_1(0) = \xi_{10}^*$, $\xi_2(0) = \xi_{20}^*$, $\xi_3(0) = \xi_{30}^*$, $\eta_1(0) = \eta_{10}^*$, $\eta_2(0) = \eta_{20}^*$ and $\eta_3(0) = \eta_{30}^*$ that satisfies $K = 0$.*

(b) *For some $p, q, \ell \in \mathbb{N}$ with greatest common divisor (g.c.d.) 1, let $\alpha = p^{2/3}\nu_1 + q^{2/3}\nu_2 + \ell^{2/3}\nu_3$, $h_1 = hp^{2/3}\nu_1/\alpha$, $h_2 = hq^{2/3}\nu_2/\alpha$ and $h_3 = h\ell^{2/3}\nu_3/\alpha$. Then $\varphi(s)$ is a periodic solution of* (4).

(c) *Assume that $s(t)$ is given by the inverse function of the function $t = \int_0^s 4\xi_1^2(\rho)\xi_2^2(\rho)\xi_3^2(\rho)\,d\rho$. Under the hypotheses of statement* (b)*, the period and the quarter of the period of the periodic solution $\varphi(s)$ using the different times t, σ, τ, υ and s is given in Table* 1.

Proof. Statement (a) follows easily from the definitions of $(\xi_1(\sigma),\eta_1(\sigma))$, $(\xi_2(\tau),\eta_2(\tau))$ and $(\xi_3(\upsilon),\eta_3(\upsilon))$ together with the definitions of $\sigma(s)$, $\tau(s)$ and $\upsilon(s)$.

We have seen that, in the time t, $(\xi_1(\sigma),\eta_1(\sigma))$, $(\xi_2(\tau),\eta_2(\tau))$ and $(\xi_3(\upsilon),\eta_3(\upsilon))$ are periodic solutions of periods $T_1(h_1,\nu_1)$, $T_2(h_2,\nu_2)$ and $T_3(h_3,\nu_3)$ respectively. Thus, in order to have a periodic solution of (4) we need that

$$p\,T_1(h_1,\nu_1) = q\,T_2(h_2,\nu_2) = \ell\,T_3(h_3,\nu_3)\,,$$

for some $p, q, \ell \in \mathbb{N}$ with g.c.d. 1. Solving equation $p\,T_1(h_1, \nu_1) = \ell\,T_3(h_3, \nu_3)$ with respect to h_1, equation $q\,T_2(h_2, \nu_2) = \ell\,T_3(h_3, \nu_3)$ with respect to h_2, and finally equation $h = h_1 + h_2 + h_3$ with respect to h_3 for the values of h_1 and h_2 calculated previously, we get that $h_1 = hp^{2/3}\nu_1/(p^{2/3}\nu_1 + q^{2/3}\nu_2 + \ell^{2/3}\nu_3)$, $h_2 = hq^{2/3}\nu_1/(p^{2/3}\nu_1 + q^{2/3}\nu_2 + \ell^{2/3}\nu_3)$ and $h_3 = h\ell^{2/3}\nu_3/(p^{2/3}\nu_1 + q^{2/3}\nu_2 + \ell^{2/3}\nu_3)$. So, statement (b) is proved.

Now we see that the time $t = T/4$ corresponds to the time $\sigma = \sigma^*/4$. In a similar way we can see that the time $t = T/4$ corresponds to the time $\tau = \tau^*/4$, $\upsilon = \upsilon^*/4$ and $s = S^*/4$.

We note that system (6) is invariant under symmetry $(\xi_1, \eta_1, \sigma) \longrightarrow (-\xi_1, \eta_1, -\sigma)$. This means that $\xi_1(\sigma) = -\xi_1(-\sigma)$. So $\xi_1^2(\sigma)$ is an even function. On the other hand, it is easy to see that $\xi_1^2(\sigma)$ is a periodic function of period $\overline{\sigma}/2$. Then, from (5), we have that

$$T_1 = \int_0^{\overline{\sigma}} 4\xi_1^2(\sigma)\, d\sigma = 2\int_0^{\overline{\sigma}/2} 4\xi_1^2(\sigma)\, d\sigma \;= 4\int_0^{\overline{\sigma}/4} 4\xi_1^2(\sigma)\, d\sigma\,.$$

Moreover, it is clear that

$$\int_0^{\overline{\sigma}/4} 4\xi_1^2(\sigma)\, d\sigma = \int_{\overline{\sigma}/4}^{\overline{\sigma}/2} 4\xi_1^2(\sigma)\, d\sigma = \frac{T_1}{4}\,.$$

Consequently

$$t(\sigma^*/4) = \int_0^{p\overline{\sigma}/4} 4\xi_1^2(\sigma)\, d\sigma = p\int_0^{\overline{\sigma}/4} 4\xi_1^2(\sigma)\, d\sigma = p\frac{T_1}{4} = \frac{T}{4}\,.$$

Therefore the time $t = T/4$ corresponds to $\sigma = \sigma^*/4$. The rest of statement (c) follows in a similar way. □

We remark that the number p in Proposition 4 represents the number of triple collisions between m_0, m_1 and m_2 during a period, the number q represents the number of triple collisions between m_0, m_3 and m_4, and the number ℓ represents the number of triple collisions between m_0, m_5 and m_6 .

We are interested in symmetric periodic solutions of (4) satisfying the energy relation $K = 0$ with $h = h_1 + h_2 + h_3$. In the next proposition we give initial conditions for those symmetric periodic solutions.

Proposition 5. *The following statements hold.*

(a) *If p is odd, and q and ℓ are even, then the solution $\varphi(s)$ given by Proposition* 4 *with initial conditions $\xi_{10}^* = \sqrt{-2\nu_1/h_1}$, $\xi_{20}^* = \sqrt{-2\nu_2/h_2}$, $\xi_{30}^* = \sqrt{-2\nu_3/h_3}$, $\eta_{10}^* = 0$, $\eta_{20}^* = 0$ and $\eta_{30}^* = 0$; is a S_{12}-symmetric periodic solution.*

(b) *If q is odd, and p and ℓ are even, then the solution $\varphi(s)$ given by Proposition* 4 *with initial conditions $\xi_{10}^* = \sqrt{-2\nu_1/h_1}$, $\xi_{20}^* = \sqrt{-2\nu_2/h_2}$, $\xi_{30}^* = \sqrt{-2\nu_3/h_3}$, $\eta_{10}^* = 0$, $\eta_{20}^* = 0$ and $\eta_{30}^* = 0$; is a S_{13}-symmetric periodic solution.*

(c) *If ℓ is odd, and p and q are even, then the solution $\varphi(s)$ given by Proposition* 4 *with initial conditions $\xi_{10}^* = \sqrt{-2\nu_1/h_1}$, $\xi_{20}^* = \sqrt{-2\nu_2/h_2}$, $\xi_{30}^* = \sqrt{-2\nu_3/h_3}$, $\eta_{10}^* = 0$, $\eta_{20}^* = 0$ and $\eta_{30}^* = 0$; is a S_{14}-symmetric periodic solution.*

(d) *If ℓ is even, and p and q are odd, then the solution $\varphi(s)$ given by Proposition* 4 *with initial conditions $\xi_{10}^* = 0$, $\xi_{20}^* = \sqrt{-2\nu_2/h_2}$, $\xi_{30}^* = \sqrt{-2\nu_3/h_3}$, $\eta_{10}^* = 4\sqrt{2}\nu_1$, $\eta_{20}^* = 0$ and $\eta_{30}^* = 0$; is a S_{23}-symmetric periodic solution.*

(e) *If q is even, and p and ℓ are odd, then the solution $\varphi(s)$ given by Proposition* 4 *with initial conditions $\xi_{10}^* = 0$, $\xi_{20}^* = \sqrt{-2\nu_2/h_2}$, $\xi_{30}^* = \sqrt{-2\nu_3/h_3}$, $\eta_{10}^* = 4\sqrt{2}\nu_1$, $\eta_{20}^* = 0$ and $\eta_{30}^* = 0$; is a S_{24}-symmetric periodic solution.*

(f) *If p is even, and q and ℓ are odd, then the solution $\varphi(s)$ given by Proposition* 4 *with initial conditions $\xi_{10}^* = \sqrt{-2\nu_1/h_1}$, $\xi_{20}^* = 0$, $\xi_{30}^* = \sqrt{-2\nu_3/h_3}$, $\eta_{10}^* = 0$, $\eta_{20}^* = 4\sqrt{2}\nu_2$ and $\eta_{30}^* = 0$; is a S_{34}-symmetric periodic solution.*

(g) *All symmetric periodic solutions of* (4) *are double symmetric periodic solutions.*

Proof. Assume that $\varphi(s)$ is a S_1-symmetric periodic solution of (4) satisfying the energy relation $K = 0$. By Proposition 1(a), $\varphi(s)$ has initial conditions $\eta_1(0) = 0$, $\eta_2(0) = 0$ and $\eta_3(0) = 0$. Moreover, the initial conditions must verify the energy relation $K = 0$; that is they must verify equations $K_i = 0$ for $i = 1, 2, 3$. Then $\xi_i(0) = \sqrt{-2\nu_i/h_i}$ for $i = 1, 2, 3$. Notice that we have only considered the positive determination in the squareroot due to the fact that the Levi–Civita transformation duplicates the orbits. We evaluate the solution $\varphi(s) = (\xi_1(\sigma(s)), \xi_2(\tau(s)), \xi_3(\upsilon(s)), \eta_1(\sigma(s)), \eta_2(\tau(s)), \eta_3(\upsilon(s)))$ at time $s = S^*/4$. We note that by Table 1, we have that $\varphi(S^*/4) = (\xi_1(p\,\overline{\sigma}/4), \xi_2(q\,\overline{\tau}/4), \xi_3(\ell\,\overline{\upsilon}/4), \eta_1(p\,\overline{\sigma}/4), \eta_2(q\,\overline{\tau}/4), \eta_3(\ell\,\overline{\upsilon}/4))$. After some computations using the explicit solutions (8), (12) and (16) we see that if p is odd and q and ℓ are even, then $\xi_1(S^*/4) = \eta_2(S^*/4) = \eta_3(S^*/4) = 0$, so $\varphi(s)$ is a S_{12}-symmetric periodic solution. If q is odd and p and ℓ are even, then $\xi_2(S^*/4) = \eta_1(S^*/4) = \eta_3(S^*/4) = 0$, so $\varphi(s)$ is a S_{13}-symmetric periodic solution. If ℓ is odd and p and q are even, then $\xi_3(S^*/4) = \eta_1(S^*/4) = \eta_2(S^*/4) = 0$, so $\varphi(s)$ is a S_{14}-symmetric periodic solution. If p is even and q and ℓ are odd, then $\xi_2(S^*/4) = \xi_3(S^*/4) = 0$. If q is even and p and ℓ are odd, then $\xi_1(S^*/4) = \xi_3(S^*/4) = 0$. If ℓ is even and p and q are odd, then $\xi_1(S^*/4) = \xi_2(S^*/4) = 0$. The last three cases are not considered because they correspond to quintuple collision orbits. Finally if p, q and ℓ are odd, then $\xi_1(S^*/4) = \xi_2(S^*/4) = \xi_3(S^*/4) = 0$. This case is not considered because it corresponds to a total collision orbit. This completes the proof of statements (a), (b) and (c).

Doing similar arguments for S_i-symmetric periodic solutions for $i = 2, 3, 4$ we can prove the remaining statements (d), (e) and (f). Finally from the proofs of all statements (a)–(f) if follows statement (g). □

5. Continuation of symmetric periodic solutions

In this section using the continuation method of Poincaré (see for instance [8]) we shall continue the symmetric periodic orbits of the octahedral 7-body problem (3) from $\mu = 0$ to symmetric periodic orbits of (3) for $\mu > 0$ sufficiently small.

5.1. The S_{12}-symmetric periodic solutions

We denote by $\varphi(s;\xi_{10},\xi_{20},\xi_{30},0,0,0,\mu)=(\xi_1(s;\xi_{10},\xi_{20},\xi_{30},\mu),\ \xi_2(s;\xi_{10},\xi_{20},\ \xi_{30},\ \mu),\ \xi_3(s;\xi_{10},\xi_{20},\xi_{30},\mu),\ \eta_1(s;\xi_{10},\xi_{20},\xi_{30},\mu),\ \eta_2(s;\xi_{10},\xi_{20},\xi_{30},\mu),\ \eta_3(s;\xi_{10},\xi_{20},\ \xi_{30},\mu))$ the solution of (3), for fixed values of $\nu_1>0$, $\nu_2>0$, $\nu_3>0$ and $h<0$, with initial conditions $\xi_1(0)=\xi_{10}$, $\xi_2(0)=\xi_{20}$, $\xi_3(0)=\xi_{30}$, $\eta_1(0)=0$, $\eta_2(0)=0$ and $\eta_3(0)=0$. From Proposition 2(a), $\varphi(s;\xi_{10},\xi_{20},\xi_{30},0,0,0,\mu)$ is a S_{12}-symmetric periodic solution of the octahedral 7-body problem with period S satisfying the energy relation $K=0$ if and only if

$$\begin{aligned}\xi_1(S/4;\xi_{10},\xi_{20},\xi_{30},\mu)&=0\,, & \eta_2(S/4;\xi_{10},\xi_{20},\xi_{30},\mu)&=0\,,\\ \eta_3(S/4;\xi_{10},\xi_{20},\xi_{30},\mu)&=0\,, & K(\xi_{10},\xi_{20},\xi_{30},\mu)&=0\,.\end{aligned}$$

We solve equation $K(\xi_{10},\xi_{20},\xi_{30},\mu)=0$ with respect to the variable ξ_{30} obtaining in this way $\xi_{30}=\widetilde{\xi_{30}}=\xi_{30}(\xi_{10},\xi_{20},\mu)$. In particular,

$$\xi_{30}(\xi_{10},\xi_{20},0)=\sqrt{2\nu_3}\xi_{10}\xi_{20}\Big/\sqrt{-2\nu_2\xi_{10}^2-2\nu_1\xi_{20}^2-h\xi_{10}^2\xi_{20}^2}\,.$$

So $\varphi(s;\xi_{10},\xi_{20},\widetilde{\xi_{30}},0,0,0,\mu)$ is a S_{12}-symmetric periodic solution of the octahedral 7-body problem with period S satisfying the energy relation $K=0$ if and only if

$$\xi_1(S/4;\xi_{10},\xi_{20},\mu)=0\,,\quad \eta_2(S/4;\xi_{10},\xi_{20},\mu)=0\,,\quad \eta_3(S/4;\xi_{10},\xi_{20},\mu)=0\,. \tag{17}$$

Notice that we have omitted the dependence with respect to ξ_{30}, which is given by the function $\xi_{30}(\xi_{10},\xi_{20},\mu)$.

Assume that $p=2p_1-1$, $q=2q_1$ and $\ell=2\ell_1$ for some $p_1,q_1,\ell_1\in\mathbb{N}$. Assume that $h_1^*=hp^{2/3}\nu_1/\alpha$, $h_2^*=hq^{2/3}\nu_1/\alpha$ and $h_3^*=h\ell^{2/3}\nu_3/\alpha$, where $\alpha=p^{2/3}\nu_1+q^{2/3}\nu_2+\ell^{2/3}\nu_3$. By Propositions 4 and 5(a), we see that $S=S^*=s(pT_1(h_1^*))=s(qT_2(h_2^*))=s(\ell T_3(h_3^*))$, $\xi_{10}=\xi_{10}^*=\sqrt{-2\nu_1/h_1^*}$ and $\xi_{20}=\xi_{20}^*=\sqrt{-2\nu_2/h_2^*}$ is a solution of (17) for $\mu=0$. This solution correspond to the known S_{12}-symmetric periodic solution $\varphi(s;\xi_{10}^*,\xi_{20}^*,\xi_{30}^*,0,0,0,\mu)$ of (3), for $\mu=0$ where $\xi_{30}^*=\sqrt{-2\nu_3/h_3^*}$. Our aim is to continue this solution of (17) from $\mu=0$ to $\mu>0$ sufficiently small.

Applying the Implicit Function Theorem to system (17) in a neighborhood of the known solution we have that if

$$\left.\begin{vmatrix}\dfrac{\partial\xi_1}{\partial s} & \dfrac{\partial\xi_1}{\partial\xi_{10}} & \dfrac{\partial\xi_1}{\partial\xi_{20}}\\[2mm] \dfrac{\partial\eta_2}{\partial s} & \dfrac{\partial\eta_2}{\partial\xi_{10}} & \dfrac{\partial\eta_2}{\partial\xi_{20}}\\[2mm] \dfrac{\partial\eta_3}{\partial s} & \dfrac{\partial\eta_3}{\partial\xi_{10}} & \dfrac{\partial\eta_3}{\partial\xi_{20}}\end{vmatrix}\right|_{\substack{s=S^*/4\\ \xi_{10}=\xi_{10}^*\\ \xi_{20}=\xi_{20}^*\\ \mu=0}}\neq 0\,, \tag{18}$$

then we can find unique analytic functions $\xi_{10}=\xi_{10}(\mu)$, $\xi_{20}=\xi_{20}(\mu)$ and $S=S(\mu)$ defined for $\mu\geq 0$ sufficiently small such that

(i) $\xi_{10}(0)=\xi_{10}^*$, $\xi_{20}(0)=\xi_{20}^*$, $S(0)=S^*$,

(ii) $\varphi(s;\xi_{10}(\mu),\xi_{20}(\mu),\widetilde{\xi_{30}},0,0,0,\mu)$ is a S_{12}-symmetric periodic solution of (3) with period $S=S(\mu)$ that satisfies the energy relation $K=0$.

The derivatives $\partial\xi_1/\partial s$, $\partial\eta_2/\partial s$ and $\partial\eta_3/\partial s$ evaluated at $s = S^*/4$, $\xi_{10} = \xi_{10}^*$, $\xi_{20} = \xi_{20}^*$ and $\mu = 0$ can be obtained evaluating the right hand side of system (3) with $\mu = 0$ (i.e., system (4)) on the solution $\varphi(s; \xi_{10}^*, \xi_{20}^*, \xi_{30}^*, 0, 0, 0, 0)$ with $s = S^*/4$. Then after some computations we get

$$\left.\frac{\partial \xi_1}{\partial s}\right|_{\substack{s=S^*/4\\ \xi_{10}=\xi_{10}^*\\ \xi_{20}=\xi_{20}^*\\ \mu=0}} = \left.\frac{\eta_1\xi_2^2\xi_3^2}{2\nu_1}\right|_{\substack{s=S^*/4\\ \xi_{10}=\xi_{10}^*\\ \xi_{20}=\xi_{20}^*\\ \mu=0}} = \frac{8\sqrt{2}(-1)^{p_1}\alpha^2}{h^2\ell^{2/3}q^{2/3}} \neq 0\,,$$

and

$$\left.\frac{\partial \eta_2}{\partial s}\right|_{\substack{s=S^*/4\\ \xi_{10}=\xi_{10}^*\\ \xi_{20}=\xi_{20}^*\\ \mu=0}} = \left.\frac{\partial \eta_3}{\partial s}\right|_{\substack{s=S^*/4\\ \xi_{10}=\xi_{10}^*\\ \xi_{20}=\xi_{20}^*\\ \mu=0}} = 0\,.$$

It only remains to compute the value of the determinant

$$\left.\begin{vmatrix} \dfrac{\partial\eta_2}{\partial\xi_{10}} & \dfrac{\partial\eta_2}{\partial\xi_{20}} \\ \dfrac{\partial\eta_3}{\partial\xi_{10}} & \dfrac{\partial\eta_3}{\partial\xi_{20}} \end{vmatrix}\right|_{\substack{s=S^*/4\\ \xi_{10}=\xi_{10}^*\\ \xi_{20}=\xi_{20}^*\\ \mu=0}} . \tag{19}$$

The value of $\partial\eta_2/\partial\xi_{10}$ evaluated at $s = S^*/4$, $\xi_{10} = \xi_{10}^*$, $\xi_{20} = \xi_{20}^*$, $\mu = 0$ is given by the derivative of the component $\eta_2(\tau(s); \xi_{10}, \xi_{20}, \xi_{30}(\xi_{10}, \xi_{20}, 0), 0, 0, 0, 0)$ of the solution of (4) with respect to ξ_{10}. Recall that we only consider solutions of (4) satisfying the energy relation $K = 0$. Since $\tau(s)$ depends on the initial condition ξ_{10} we have that

$$\left.\frac{\partial \eta_2}{\partial \xi_{10}}\right|_{\substack{s=S^*/4\\ \xi_{10}=\xi_{10}^*\\ \xi_{20}=\xi_{20}^*\\ \mu=0}} = \left.\left(\frac{\partial\eta_2}{\partial\tau}\frac{\partial\tau}{\partial\xi_{10}} + \frac{\partial\eta_2}{\partial\overline{\xi_{10}}} + \frac{\partial\eta_2}{\partial\xi_{30}}\frac{\partial\xi_{30}}{\partial\xi_{10}}\right)\right|_{\substack{s=S^*/4\\ \xi_{10}=\xi_{10}^*\\ \xi_{20}=\xi_{20}^*\\ \mu=0}} , \tag{20}$$

where $\partial\eta_2/\partial\overline{\xi_{10}}$ denotes the partial of η_2 with respect to the second component.

From (5), (9) and (13) we have that the times σ, τ and υ are related by the following equations

$$\xi_1^2(\sigma)\,d\sigma = \xi_2^2(\tau)\,d\tau\,, \quad \xi_1^2(\sigma)\,d\sigma = \xi_3^2(\upsilon)\,d\upsilon\,, \quad \xi_2^2(\tau)\,d\tau = \xi_3^2(\upsilon)\,d\upsilon\,.$$

Integrating these three equations over the solutions (8), (12), and (16) with initial conditions $\xi_1(0) = \xi_{10}$, $\xi_2(0) = \xi_{20}$, $\xi_{30} = \xi_{30}$, $\eta_1(0) = \eta_2(0) = \eta_3(0) = 0$ and assuming that $\sigma(0) = \tau(0) = \upsilon(0) = 0$, we have that $\sigma(s)$, $\tau(s)$ and $\upsilon(s)$ are related by the system of equations

$$\begin{aligned} &\xi_{10}^2\left(\frac{\sigma(s)}{2} + \frac{\sin(2\sigma(s)w_1)}{4w_1}\right) - \xi_{20}^2\left(\frac{\tau(s)}{2} + \frac{\sin(2\tau(s)w_2)}{4w_2}\right) = 0\,,\\ &\xi_{10}^2\left(\frac{\sigma(s)}{2} + \frac{\sin(2\sigma(s)w_1)}{4w_1}\right) - \xi_{30}^2\left(\frac{\upsilon(s)}{2} + \frac{\sin(2\upsilon(s)w_3)}{4w_3}\right) = 0\,,\\ &\xi_{20}^2\left(\frac{\tau(s)}{2} + \frac{\sin(2\tau(s)w_2)}{4w_2}\right) - \xi_{30}^2\left(\frac{\upsilon(s)}{2} + \frac{\sin(2\upsilon(s)w_3)}{4w_3}\right) = 0\,, \end{aligned} \tag{21}$$

where $\xi_{30} = \sqrt{2\nu_3}\xi_{10}\xi_{20}/\sqrt{-2\nu_2\xi_{10}^2 - 2\nu_1\xi_{20}^2 - h\xi_{10}^2\xi_{20}^2}$ and

$$\begin{aligned} w_1 &= 2\sqrt{-h_1/\nu_1}\,, \quad \text{with} \quad h_1 = -2\nu_1/\xi_{10}^2\,, \\ w_2 &= 2\sqrt{-h_2/\nu_2}\,, \quad \text{with} \quad h_2 = -2\nu_2/\xi_{20}^2\,, \\ w_3 &= 2\sqrt{-h_3/\nu_3}\,, \quad \text{with} \quad h_3 = h - h_1 - h_2\,. \end{aligned} \tag{22}$$

The last conditions come from the fact that we only consider solutions of (4) satisfying the energy relation $K = 0$.

Derivating the three equations of (21) with respect to ξ_{10} we obtain a linear system of equations with $\partial\sigma/\partial\xi_{10}$, $\partial\tau/\partial\xi_{10}$ and $\partial\upsilon/\partial\xi_{10}$ as unknowns. With the help of Mathematica we solve this system with respect to the variables $\partial\tau/\partial\xi_{10}$ and $\partial\upsilon/\partial\xi_{10}$ and we evaluate the solution at $s = S^*/4$, $\xi_{10} = \xi_{10}^*$ and $\xi_{20} = \xi_{20}^*$. After hard simplifications we obtain

$$\left.\frac{\partial\tau}{\partial\xi_{10}}\right|_{\substack{s = S^*/4 \\ \xi_{10} = \xi_{10}^* \\ \xi_{20} = \xi_{20}^* \\ \mu = 0}} = \frac{3\pi p^{1/3} q^{2/3}}{8\sqrt{2}}\,, \qquad \left.\frac{\partial\upsilon}{\partial\xi_{10}}\right|_{\substack{s = S^*/4 \\ \xi_{10} = \xi_{10}^* \\ \xi_{20} = \xi_{20}^* \\ \mu = 0}} = \frac{\pi(p\nu_1 + 3\ell^{2/3}p^{1/3}\nu_3)}{8\sqrt{2}\nu_3}\,.$$

The derivative $\partial\eta_2/\partial\tau$ evaluated at $s = S^*/4$, $\xi_{10} = \xi_{10}^*$, $\xi_{20} = \xi_{20}^*$, $\mu = 0$ can be obtained evaluating the right hand side of the second equation of (10) at the solution (12) with $h_2 = h_2^*$, $\tau = q\,2\pi/(4w_2)$, $w_2 = 2\sqrt{-h_2^*/\nu_2}$, $\xi_{20} = \xi_{20}^*$ and $\eta_{20} = 0$. After some simplifications we get

$$\left.\frac{\partial\eta_2}{\partial\tau}\right|_{\substack{s = S^*/4 \\ \xi_{10} = \xi_{10}^* \\ \xi_{20} = \xi_{20}^* \\ \mu = 0}} = 8(-1)^{q_1+1}\sqrt{2}\,\nu_2\, q^{1/3}\sqrt{\frac{-h}{\alpha}}\,,$$

Finally, since the second equation of (12) does not depend on ξ_{10} and ξ_{30}, we have that

$$\left.\frac{\partial\eta_2}{\partial\xi_{10}}\right|_{\substack{s = S^*/4 \\ \xi_{10} = \xi_{10}^* \\ \xi_{20} = \xi_{20}^* \\ \mu = 0}} = \left.\frac{\partial\eta_2}{\partial\xi_{30}}\right|_{\substack{s = S^*/4 \\ \xi_{10} = \xi_{10}^* \\ \xi_{20} = \xi_{20}^* \\ \mu = 0}} = 0\,.$$

So (20) becomes

$$\left.\frac{\partial\eta_2}{\partial\xi_{10}}\right|_{\substack{s = S^*/4 \\ \xi_{10} = \xi_{10}^* \\ \xi_{20} = \xi_{20}^* \\ \mu = 0}} = (-1)^{q_1}\,\nu_2\, 3\pi p^{1/3} q\,\sqrt{\frac{-h}{\alpha}}\,.$$

We compute the values of $\partial\eta_2/\partial\xi_{20}$, $\partial\eta_3/\partial\xi_{10}$ and $\partial\eta_3/\partial\xi_{20}$ in a similar way and after several simplifications we see that the value of determinant (19) is $9(-1)^{l_1+q_1}\pi^2\nu_2 h p^{1/3} q^{4/3}\ell^{1/3}$. Therefore (18) is different form zero. In particular, (18) becomes $-72(-1)^{p_1+q_1+\ell_1}\sqrt{2}\pi^2 p^{1/3} q^{2/3}\nu_2\alpha^2/(h\ell^{1/3})$. In short, we have proved the following result.

Theorem 6. *Given $\nu_1 > 0$, $\nu_2 > 0$, $\nu_3 > 0$, $h < 0$, p an odd positive integer and q and ℓ even positive integers, the S_{12}-symmetric periodic solution of the octahedral 7-body problem (3) for $\mu = 0$ with initial conditions $\xi_1(0) = \sqrt{-2\nu_1/h_1^*}$,*

$\xi_2(0) = \sqrt{-2\nu_2/h_2^*}$, $\xi_3(0) = \sqrt{-2\nu_3/h_3^*}$, $\eta_1(0) = 0$, $\eta_2(0) = 0$ and $\eta_3(0) = 0$, can be continued to a μ-parameter family of S_{12}-symmetric periodic orbits of the octahedral 7-body problem (3) for $\mu > 0$ sufficiently small. Here $h_1^* = hp^{2/3}\nu_1/\alpha$, $h_2^* = hq^{2/3}\nu_1/\alpha$ and $h_3^* = h\ell^{2/3}\nu_3/\alpha$ where $\alpha = p^{2/3}\nu_1 + q^{2/3}\nu_2 + \ell^{2/3}\nu_3$.

5.2. The S_{13}-symmetric periodic solutions

We consider the S_{13}-symmetric periodic solution $\varphi(s;\xi_{10}^*,\xi_{20}^*,\xi_{30}^*,0,0,0,\mu)$ of (3) for $\mu = 0$ with period $S = S^* = s(p\,T_1(h_1^*)) = s(q\,T_2(h_2^*)) = s(\ell\,T_3(h_3^*))$. Here $\xi_{10}^* = \sqrt{-2\nu_1/h_1^*}$, $\xi_{20}^* = \sqrt{-2\nu_2/h_2^*}$, $\xi_{30}^* = \sqrt{-2\nu_3/h_3^*}$, $p = 2p_1$, $q = 2q_1 - 1$, $\ell = 2\ell_1$ for some $p_1, q_1, \ell_1 \in \mathbb{N}$, and finally $h_1^* = hp^{2/3}\nu_1/\alpha$, $h_2^* = hq^{2/3}\nu_1/\alpha$ and $h_3^* = h\ell^{2/3}\nu_3/\alpha$ where $\alpha = p^{2/3}\nu_1 + q^{2/3}\nu_2 + \ell^{2/3}\nu_3$, see Propositions 4 and 5(b). Now we proceed as in Subsection 5.1 and we obtain that the solution $\varphi(s;\xi_{10}^*,\xi_{20}^*,\xi_{30}^*,0,0,0,\mu)$ can be continued to a family of S_{13}-symmetric periodic solutions $\varphi(s;\xi_{10}(\mu),\xi_{20}(\mu),\widetilde{\xi_{30}},0,0,0,\mu)$ of (3) for $\mu > 0$ with period $S = S(\mu)$ that satisfies the energy relation $K = 0$ because

$$\left|\begin{array}{ccc} \dfrac{\partial\xi_2}{\partial s} & \dfrac{\partial\xi_2}{\partial\xi_{10}} & \dfrac{\partial\xi_2}{\partial\xi_{20}} \\ \dfrac{\partial\eta_1}{\partial s} & \dfrac{\partial\eta_1}{\partial\xi_{10}} & \dfrac{\partial\eta_1}{\partial\xi_{20}} \\ \dfrac{\partial\eta_3}{\partial s} & \dfrac{\partial\eta_3}{\partial\xi_{10}} & \dfrac{\partial\eta_3}{\partial\xi_{20}} \end{array}\right|_{\substack{s = S^*/4 \\ \xi_{10} = \xi_{10}^* \\ \xi_{20} = \xi_{20}^* \\ \mu = 0}} = \frac{72(-1)^{p_1+q_1+\ell_1}\sqrt{2}\pi^2 p^{2/3} q^{1/3}\nu_1\alpha^2}{h\ell^{1/3}} \neq 0\,.$$

Hence the following result is proved.

Theorem 7. *Given $\nu_1 > 0$, $\nu_2 > 0$, $\nu_3 > 0$, $h < 0$, q an odd positive integer and p and ℓ even positive integers, the S_{13}-symmetric periodic solution of the octahedral 7-body problem (3) for $\mu = 0$ with initial conditions $\xi_1(0) = \sqrt{-2\nu_1/h_1^*}$, $\xi_2(0) = \sqrt{-2\nu_2/h_2^*}$, $\xi_3(0) = \sqrt{-2\nu_3/h_3^*}$, $\eta_1(0) = 0$, $\eta_2(0) = 0$ and $\eta_3(0) = 0$, can be continued to a μ-parameter family of S_{13}-symmetric periodic orbits of the octahedral 7-body problem (3) for $\mu > 0$ sufficiently small. Here $h_1^* = hp^{2/3}\nu_1/\alpha$, $h_2^* = hq^{2/3}\nu_1/\alpha$ and $h_3^* = h\ell^{2/3}\nu_3/\alpha$ where $\alpha = p^{2/3}\nu_1 + q^{2/3}\nu_2 + \ell^{2/3}\nu_3$.*

5.3. The S_{14}-symmetric periodic solutions

We consider the S_{14}-symmetric periodic solution $\varphi(s;\xi_{10}^*,\xi_{20}^*,\xi_{30}^*,0,0,0,\mu)$ of (3) for $\mu = 0$ with period $S = S^* = s(p\,T_1(h_1^*)) = s(q\,T_2(h_2^*)) = s(\ell\,T_3(h_3^*))$. Here $\xi_{10}^* = \sqrt{-2\nu_1/h_1^*}$, $\xi_{20}^* = \sqrt{-2\nu_2/h_2^*}$, $\xi_{30}^* = \sqrt{-2\nu_3/h_3^*}$, $p = 2p_1$, $q = 2q_1$, $\ell = 2\ell_1 - 1$ for some $p_1, q_1, \ell_1 \in \mathbb{N}$, and finally $h_1^* = hp^{2/3}\nu_1/\alpha$, $h_2^* = hq^{2/3}\nu_1/\alpha$ and $h_3^* = h\ell^{2/3}\nu_3/\alpha$ where $\alpha = p^{2/3}\nu_1 + q^{2/3}\nu_2 + \ell^{2/3}\nu_3$, see Propositions 4 and 5(c). Now we proceed as in Subsection 5.1 and we obtain that the solution $\varphi(s;\xi_{10}^*,\xi_{20}^*,\xi_{30}^*,0,0,0,\mu)$ can be continued to a family of S_{14}-symmetric periodic solutions $\varphi(s;\xi_{10}(\mu),\xi_{20}(\mu),\widetilde{\xi_{30}},0,0,0,\mu)$ of (3) for $\mu > 0$ with period $S = S(\mu)$

that satisfies the energy relation $K = 0$ because

$$\left|\begin{array}{ccc} \dfrac{\partial \xi_3}{\partial s} & \dfrac{\partial \xi_3}{\partial \xi_{10}} & \dfrac{\partial \xi_3}{\partial \xi_{20}} \\ \dfrac{\partial \eta_1}{\partial s} & \dfrac{\partial \eta_1}{\partial \xi_{10}} & \dfrac{\partial \eta_1}{\partial \xi_{20}} \\ \dfrac{\partial \eta_2}{\partial s} & \dfrac{\partial \eta_2}{\partial \xi_{10}} & \dfrac{\partial \eta_2}{\partial \xi_{20}} \end{array}\right|_{\substack{s = S^*/4 \\ \xi_{10} = \xi_{10}^* \\ \xi_{20} = \xi_{20}^* \\ \mu = 0}} = -\frac{72(-1)^{p_1+q_1+\ell_1}\sqrt{2}\pi^2 p^{2/3} q^{2/3} \nu_1 \nu_2 \alpha^2}{\ell^{2/3} h \nu_3} \neq 0\,.$$

Hence the following result is proved.

Theorem 8. *Given $\nu_1 > 0$, $\nu_2 > 0$, $\nu_3 > 0$, $h < 0$, ℓ an odd positive integer and p and q even positive integers, the S_{14}-symmetric periodic solution of the octahedral 7-body problem (3) for $\mu = 0$ with initial conditions $\xi_1(0) = \sqrt{-2\nu_1/h_1^*}$, $\xi_2(0) = \sqrt{-2\nu_2/h_2^*}$, $\xi_3(0) = \sqrt{-2\nu_3/h_3^*}$, $\eta_1(0) = 0$, $\eta_2(0) = 0$ and $\eta_3(0) = 0$, can be continued to a μ-parameter family of S_{14}-symmetric periodic orbits of the octahedral 7-body problem (3) for $\mu > 0$ sufficiently small. Here $h_1^* = hp^{2/3}\nu_1/\alpha$, $h_2^* = hq^{2/3}\nu_1/\alpha$ and $h_3^* = h\ell^{2/3}\nu_3/\alpha$ where $\alpha = p^{2/3}\nu_1 + q^{2/3}\nu_2 + \ell^{2/3}\nu_3$.*

5.4. The S_{23}-symmetric periodic solutions

We denote by $\varphi(s; 0, \xi_{20}, \xi_{30}, \eta_{10}, 0, 0, \mu) = (\xi_1(s; \xi_{20}, \xi_{30}, \eta_{10}, \mu), \xi_2(s; \xi_{20}, \xi_{30}, \eta_{10}, \mu), \xi_3(s; \xi_{20}, \xi_{30}, \eta_{10}, \mu), \eta_1(s; \xi_{20}, \xi_{30}, \eta_{10}, \mu), \eta_2(s; \xi_{20}, \xi_{30}, \eta_{10}, \mu), \eta_3(s; \xi_{20}, \xi_{30}, \eta_{10}, \mu))$ the solution of (3), for fixed values of $\nu_1 > 0$, $\nu_2 > 0$, $\nu_3 > 0$ and $h < 0$, with initial conditions $\xi_1(0) = 0$, $\xi_2(0) = \xi_{20}$, $\xi_3(0) = \xi_{30}$, $\eta_1(0) = \eta_{10}$, $\eta_2(0) = 0$ and $\eta_3(0) = 0$. From Proposition 2(d), $\varphi(s; 0, \xi_{20}, \xi_{30}, \eta_{10}, 0, 0, \mu)$ is a S_{23}-symmetric periodic solution of the octahedral 7-body problem with period S satisfying the energy relation $K = 0$ if and only if

$$\begin{aligned} &\xi_2(S/4; \xi_{20}, \xi_{30}, \eta_{10}, \mu) = 0\,, &\qquad &\eta_1(S/4; \xi_{20}, \xi_{30}, \eta_{10}, \mu) = 0\,, \\ &\eta_3(S/4; \xi_{20}, \xi_{30}, \eta_{10}, \mu) = 0\,, &\qquad &K(\xi_{20}, \xi_{30}, \eta_{10}, \mu) = 0\,. \end{aligned}$$

We solve equation $K(\xi_{20}, \xi_{30}, \eta_{10}, \mu) = 0$ with respect the variable η_{10} obtaining in this way

$$\eta_{10} = \widetilde{\eta_{10}} = 2\sqrt{2}\nu_1\sqrt{4 + \mu\nu_1}\,. \tag{23}$$

So $\varphi(s; 0, \xi_{20}, \xi_{30}, \widetilde{\eta_{10}}, 0, 0, \mu)$ is a S_{23}-symmetric periodic solution of the octahedral 7-body problem with period S satisfying the energy relation $K = 0$ if and only if

$$\xi_2(S/4; \xi_{20}, \xi_{30}, \mu) = 0\,, \quad \eta_1(S/4; \xi_{20}, \xi_{30}, \mu) = 0\,, \quad \eta_3(S/4; \xi_{20}, \xi_{30}, \mu) = 0\,. \tag{24}$$

Notice that we have omitted the dependence with respect to η_{10}, which is given by (23).

Assume that $p = 2p_1 - 1$, $q = 2q_1 - 1$ and $\ell = 2\ell_1$ for some $p_1, q_1, \ell_1 \in \mathbb{N}$. Assume that $h_1^* = hp^{2/3}\nu_1/\alpha$, $h_2^* = hq^{2/3}\nu_1/\alpha$ and $h_3^* = h\ell^{2/3}\nu_3/\alpha$ where $\alpha = p^{2/3}\nu_1 + q^{2/3}\nu_2 + \ell^{2/3}\nu_3$. By Propositions 4 and 5(d), we see that $S = S^* = s(p\,T_1(h_1^*)) = s(q\,T_2(h_2^*)) = s(\ell\,T_3(h_3^*))$, $\xi_{20} = \xi_{20}^* = \sqrt{-2\nu_2/h_2^*}$ and $\xi_{30} = \xi_{30}^* = \sqrt{-2\nu_3/h_3^*}$ is a solution of (24) for $\mu = 0$. This solution correspond to the known S_{23}-symmetric periodic solution $\varphi(s; 0, \xi_{20}^*, \xi_{30}^*, \eta_{10}^*, 0, 0, \mu)$ of (3), for $\mu = 0$ where $\eta_{10}^* = 4\sqrt{2}\nu_1$. Our aim is to continue this solution of (24) from $\mu = 0$ to $\mu > 0$ sufficiently small.

We proceed as in Subsection 5.1. Then applying the Implicit Function Theorem to system (24) in a neighborhood of the known solution and after doing the corresponding computations we see that

$$\left| \begin{array}{ccc} \dfrac{\partial \xi_2}{\partial s} & \dfrac{\partial \xi_2}{\partial \xi_{20}} & \dfrac{\partial \xi_2}{\partial \xi_{30}} \\ \dfrac{\partial \eta_1}{\partial s} & \dfrac{\partial \eta_1}{\partial \xi_{20}} & \dfrac{\partial \eta_1}{\partial \xi_{30}} \\ \dfrac{\partial \eta_3}{\partial s} & \dfrac{\partial \eta_3}{\partial \xi_{20}} & \dfrac{\partial \eta_3}{\partial \xi_{30}} \end{array} \right|_{\substack{s = S^*/4 \\ \xi_{20} = \xi_{20}^* \\ \xi_{30} = \xi_{30}^* \\ \mu = 0}} = -\frac{72(-1)^{p_1+q_1+\ell_1}\sqrt{2}\pi^2 q^{1/3}\ell^{2/3}\nu_3\alpha^2}{p^{1/3}h},$$

which is different from zero. Then we can find unique analytic functions $\xi_{20} = \xi_{20}(\mu)$, $\xi_{30} = \xi_{30}(\mu)$, and $S = S(\mu)$ defined for $\mu \geq 0$ sufficiently small, such that

(i) $\xi_{20}(0) = \xi_{20}^*$, $\xi_{30}(0) = \xi_{30}^*$, $S(0) = S^*$,

(ii) $\varphi(s; 0, \xi_{20}(\mu), \xi_{30}(\mu), \widetilde{\eta_{10}}, 0, 0, \mu)$, where $\widetilde{\eta_{10}}$ is given by (23), is a S_{23}-symmetric periodic solution of (3) with period $S = S(\mu)$ that satisfies the energy relation $K = 0$.

This proves the following result.

Theorem 9. *Given $\nu_1 > 0$, $\nu_2 > 0$, $\nu_3 > 0$, $h < 0$, ℓ an even positive integer and p and q odd positive integers, the S_{23}-symmetric periodic solution of the octahedral 7-body problem (3) for $\mu = 0$ with initial conditions $\xi_1(0) = 0$, $\xi_2(0) = \sqrt{-2\nu_2/h_2^*}$, $\xi_3(0) = \sqrt{-2\nu_3/h_3^*}$, $\eta_1(0) = 4\sqrt{2}\nu_1$, $\eta_2(0) = 0$ and $\eta_3(0) = 0$, can be continued to a μ-parameter family of S_{23}-symmetric periodic orbits of the octahedral 7-body problem (3) for $\mu > 0$ sufficiently small. Here $h_1^* = hp^{2/3}\nu_1/\alpha$, $h_2^* = hq^{2/3}\nu_1/\alpha$ and $h_3^* = h\ell^{2/3}\nu_3/\alpha$ where $\alpha = p^{2/3}\nu_1 + q^{2/3}\nu_2 + \ell^{2/3}\nu_3$.*

5.5. The S_{24}-symmetric periodic solutions

We consider the S_{24}-symmetric periodic solution $\varphi(s; 0, \xi_{20}^*, \xi_{30}^*, \eta_{10}^*, 0, 0, \mu)$ of (3) for $\mu = 0$ with period $S = S^* = s(p\,T_1(h_1^*)) = s(q\,T_2(h_2^*)) = s(\ell\,T_3(h_3^*))$. Here $\xi_{20}^* = \sqrt{-2\nu_2/h_2^*}$, $\xi_{30}^* = \sqrt{-2\nu_3/h_3^*}$, $\eta_{10}^* = 4\sqrt{2}\nu_1$, $p = 2p_1 - 1$, $q = 2q_1$, $\ell = 2\ell_1 - 1$ for some $p_1, q_1, \ell_1 \in \mathbb{N}$, and finally $h_1^* = hp^{2/3}\nu_1/\alpha$, $h_2^* = hq^{2/3}\nu_1/\alpha$ and $h_3^* = h\ell^{2/3}\nu_3/\alpha$ where $\alpha = p^{2/3}\nu_1 + q^{2/3}\nu_2 + \ell^{2/3}\nu_3$, see Propositions 4 and 5(e). Now we proceed as in Subsection 5.4 and we obtain that the solution $\varphi(s; 0, \xi_{20}^*, \xi_{30}^*, \eta_{10}^*, 0, 0, \mu)$ can be continued to a family of S_{24}-symmetric periodic solutions $\varphi(s; 0, \xi_{20}(\mu), \xi_{30}(\mu), \widetilde{\eta_{10}}, 0, 0, \mu)$ of (3) for $\mu > 0$ with period $S = S(\mu)$

that satisfies the energy relation $K = 0$ because

$$\left|\begin{matrix} \dfrac{\partial \xi_3}{\partial s} & \dfrac{\partial \xi_3}{\partial \xi_{20}} & \dfrac{\partial \xi_3}{\partial \xi_{30}} \\ \dfrac{\partial \eta_1}{\partial s} & \dfrac{\partial \eta_1}{\partial \xi_{20}} & \dfrac{\partial \eta_1}{\partial \xi_{30}} \\ \dfrac{\partial \eta_2}{\partial s} & \dfrac{\partial \eta_2}{\partial \xi_{20}} & \dfrac{\partial \eta_2}{\partial \xi_{30}} \end{matrix}\right| \Bigg|_{\substack{s = S^*/4 \\ \xi_{20} = \xi_{20}^* \\ \xi_{30} = \xi_{30}^* \\ \mu = 0}} = \frac{72(-1)^{p_1+q_1+\ell_1}\sqrt{2}\,\pi^2 q^{2/3}\ell^{1/3}\nu_2\alpha^2}{p^{1/3}h} \neq 0\,.$$

Hence the following result is proved.

Theorem 10. *Given $\nu_1 > 0$, $\nu_2 > 0$, $\nu_3 > 0$, $h < 0$, q an even positive integer and p and ℓ odd positive integers, the S_{24}-symmetric periodic solution of the octahedral 7-body problem (3) for $\mu = 0$ with initial conditions $\xi_1(0) = 0$, $\xi_2(0) = \sqrt{-2\nu_2/h_2^*}$, $\xi_3(0) = \sqrt{-2\nu_3/h_3^*}$, $\eta_1(0) = 4\sqrt{2}\nu_1$, $\eta_2(0) = 0$ and $\eta_3(0) = 0$, can be continued to a μ-parameter family of S_{24}-symmetric periodic orbits of the octahedral 7-body problem (3) for $\mu > 0$ sufficiently small. Here $h_1^* = hp^{2/3}\nu_1/\alpha$, $h_2^* = hq^{2/3}\nu_1/\alpha$ and $h_3^* = h\ell^{2/3}\nu_3/\alpha$ where $\alpha = p^{2/3}\nu_1 + q^{2/3}\nu_2 + \ell^{2/3}\nu_3$.*

5.6. The S_{34}-symmetric periodic solutions

We denote by $\varphi(s;\xi_{10},0,\xi_{30},0,\eta_{20},0,\mu) = (\xi_1(s;\xi_{10},\xi_{30},\eta_{20},\mu), \xi_2(s;\xi_{10},\xi_{30},\eta_{20},\mu), \xi_3(s;\xi_{10},\xi_{30},\eta_{20},\mu), \eta_1(s;\xi_{10},\xi_{30},\eta_{20},\mu), \eta_2(s;\xi_{10},\xi_{30},\eta_{20},\mu), \eta_3(s;\xi_{10},\xi_{30},\eta_{20},\mu))$ the solution of (3), for fixed values of $\nu_1 > 0$, $\nu_2 > 0$, $\nu_3 > 0$ and $h < 0$, with initial conditions $\xi_1(0) = \xi_{10}$, $\xi_2(0) = 0$, $\xi_3(0) = \xi_{30}$, $\eta_1(0) = 0$, $\eta_2(0) = \eta_{20}$ and $\eta_3(0) = 0$. From Proposition 2(f), $\varphi(s;\xi_{10},0,\xi_{30},0,\eta_{20},0,\mu)$ is a S_{34}-symmetric periodic solution of the octahedral 7-body problem with period S satisfying the energy relation $K = 0$ if and only if

$$\xi_3(S/4;\xi_{10},\xi_{30},\eta_{20},\mu) = 0\,, \qquad \eta_1(S/4;\xi_{10},\xi_{30},\eta_{20},\mu) = 0\,,$$
$$\eta_2(S/4;\xi_{10},\xi_{30},\eta_{20},\mu) = 0\,, \qquad K(\xi_{10},\xi_{30},\eta_{20},\mu) = 0\,.$$

We solve equation $K(\xi_{10},\xi_{30},\eta_{20},\mu) = 0$ with respect the variable η_{20} obtaining in this way

$$\eta_{20} = \widetilde{\eta_{20}} = 2\sqrt{2}\nu_2\sqrt{4+\mu\nu_2}\,. \tag{25}$$

So $\varphi(s;\xi_{10},0,\xi_{30},0,\widetilde{\eta_{20}},0,\mu)$ is a S_{34}-symmetric periodic solution of the octahedral 7-body problem with period S satisfying the energy relation $K = 0$ if and only if

$$\xi_3(S/4;\xi_{10},\xi_{30},\mu) = 0\,, \quad \eta_1(S/4;\xi_{10},\xi_{30},\mu) = 0\,, \quad \eta_2(S/4;\xi_{10},\xi_{30},\mu) = 0\,. \tag{26}$$

Notice that we have omitted the dependence with respect to η_{20}, which is given by (25).

Assume that $p = 2p_1$, $q = 2q_1 - 1$ and $\ell = 2\ell_1 - 1$ for some $p_1, q_1, \ell_1 \in \mathbb{N}$. Assume that $h_1^* = hp^{2/3}\nu_1/\alpha$, $h_2^* = hq^{2/3}\nu_1/\alpha$ and $h_3^* = h\ell^{2/3}\nu_3/\alpha$ where $\alpha = p^{2/3}\nu_1 + q^{2/3}\nu_2 + \ell^{2/3}\nu_3$. By Propositions 4 and 5(f), we see that $S = S^* = s(p\,T_1(h_1^*)) = s(q\,T_2(h_2^*)) = s(\ell\,T_3(h_3^*))$, $\xi_{10} = \xi_{10}^* = \sqrt{-2\nu_1/h_1^*}$ and $\xi_{30} = \xi_{30}^* = \sqrt{-2\nu_3/h_3^*}$ is a solution of (26) for $\mu = 0$. This solution correspond to the known S_{34}-symmetric periodic solution $\varphi(s;\xi_{10}^*,0,\xi_{30}^*,0,\eta_{20}^*,0,\mu)$ of (3), for $\mu=0$ where $\eta_{20}^*=4\sqrt{2}\nu_2$. Our aim is to continue this solution of (26) from $\mu=0$ to $\mu>0$ sufficiently small.

We proceed as in Subsection 5.1. Then applying the Implicit Function Theorem to system (26) in a neighborhood of the known solution and after doing the corresponding computations we see that

$$\left|\begin{array}{ccc} \dfrac{\partial\xi_3}{\partial s} & \dfrac{\partial\xi_3}{\partial\xi_{10}} & \dfrac{\partial\xi_3}{\partial\xi_{30}} \\ \dfrac{\partial\eta_1}{\partial s} & \dfrac{\partial\eta_1}{\partial\xi_{10}} & \dfrac{\partial\eta_1}{\partial\xi_{30}} \\ \dfrac{\partial\eta_2}{\partial s} & \dfrac{\partial\eta_2}{\partial\xi_{10}} & \dfrac{\partial\eta_2}{\partial\xi_{30}} \end{array}\right|_{\substack{s=S^*/4\\ \xi_{10}=\xi_{10}^*\\ \xi_{30}=\xi_{30}^*\\ \mu=0}} = -\frac{72(-1)^{p_1+q_1+\ell_1}\sqrt{2}\pi^2p^{2/3}\ell^{1/3}\nu_1\alpha^2}{q^{1/3}h},$$

Then we can find unique analytic functions $\xi_{10}=\xi_{10}(\mu)$, $\xi_{30}=\xi_{30}(\mu)$, and $S=S(\mu)$ defined for $\mu\geq 0$ sufficiently small, such that

(i) $\xi_{10}(0)=\xi_{10}^*$, $\xi_{30}(0)=\xi_{30}^*$, $S(0)=S^*$,

(ii) $\varphi(s;\xi_{10}(\mu),0,\xi_{30}(\mu),0,\widetilde{\eta_{20}},0,\mu)$, where $\widetilde{\eta_{20}}$ is given by (25), is a S_{34}-symmetric periodic solution of (3) with period $S=S(\mu)$ that satisfies the energy relation $K=0$.

This proves the following result.

Theorem 11. *Given $\nu_1>0$, $\nu_2>0$, $\nu_3>0$, $h<0$, p an even positive integer and q and ℓ odd positive integers, the S_{34}-symmetric periodic solution of the octahedral 7-body problem (3) for $\mu=0$ with initial conditions $\xi_1(0)=\sqrt{-2\nu_1/h_1^*}$, $\xi_2(0)=0$, $\xi_3(0)=\sqrt{-2\nu_3/h_3^*}$, $\eta_1(0)=0$, $\eta_2(0)=4\sqrt{2}\nu_2$ and $\eta_3(0)=0$, can be continued to a μ-parameter family of S_{34}-symmetric periodic orbits of the octahedral 7-body problem (3) for $\mu>0$ sufficiently small. Here $h_1^*=hp^{2/3}\nu_1/\alpha$, $h_2^*=hq^{2/3}\nu_1/\alpha$ and $h_3^*=h\ell^{2/3}\nu_3/\alpha$ where $\alpha=p^{2/3}\nu_1+q^{2/3}\nu_2+\ell^{2/3}\nu_3$.*

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