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Qualitative Theory of Dynamical Systems

On the Existence of Central Configurations of *p* Nested *n*-gons

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Abstract In this paper we prove the existence of central configurations of the pn-body problem where the masses are at the vertices of p nested regular n-gons with a common center for all $p \geq 2$ and $n \geq 2$. In such configurations all the masses on the same n-gon are equal, but masses on different n-gons could be different.

Keywords Planar central configurations · Nested regular n-gons

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1 Introduction

We consider the planar N-body problem

$$m_k\,\ddot{\mathbf{q}}_k = -\sum_{j=1,\,j\neq k}^{N} G\,m_k\,m_j\,\frac{\mathbf{q}_k-\mathbf{q}_j}{|\mathbf{q}_k-\mathbf{q}_j|^3},\quad k=1,\ldots,N,$$

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where $\mathbf{q}_k \in \mathbb{R}^2$ is the position vector of the punctual mass m_k in an inertial coordinate system, and G is the gravitational constant which can be taken equal to one by choosing conveniently the unit of time. By fixing the center of mass $\sum_{k=1}^{N} m_k\, \mathbf{q}_k / \sum_{k=1}^{N} m_k$ of the system at the origin of $\mathbb{R}^{2N}$, the *configuration space* of the planar N-body problem is

$$\mathcal{E} = \left\{ (\mathbf{q}_1, \ldots, \mathbf{q}_N) \in \mathbb{R}^{2N} \;:\; \sum_{k=1}^{N} m_k\, \mathbf{q}_k = 0,\;\; \mathbf{q}_k \neq \mathbf{q}_j,\;\; \text{for } k \neq j \right\}.$$

Given $m_1, \ldots, m_N$ a configuration $(\mathbf{q}_1, \ldots, \mathbf{q}_N) \in \mathcal{E}$ is *central* if there exists a positive constant λ such that

$$\ddot{\mathbf{q}}_k = -\lambda\, \mathbf{q}_k\,, \quad k = 1, \ldots, N.$$

Thus a central configuration $(\mathbf{q}_1, \ldots, \mathbf{q}_N) \in \mathcal{E}$ of the N-body problem with positive masses $m_1, \ldots, m_N$ is a solution of the system of equations

$$\sum_{j=1,\, j\neq i}^{N} m_j \frac{\mathbf{q}_i - \mathbf{q}_j}{|\mathbf{q}_i - \mathbf{q}_j|^3} = \lambda\, \mathbf{q}_i\,, \quad i = 1, \ldots, N, \tag{1.1}$$

for some λ. System (1.1) can be written as $\partial U/\partial \mathbf{q} + \lambda \partial I/\partial \mathbf{q} = 0$ where

$$U = \sum_{1\leq i<j\leq N} \frac{m_i\, m_j}{|\mathbf{q}_i - \mathbf{q}_j|}, \quad \text{and} \quad I = \frac{1}{2}\sum_{i=1}^{N} m_i |\mathbf{q}_i|^2,$$

are the potential and the moment of inertia of the problem, respectively. Then by the Euler's Theorem on homogeneous functions (see for instance [6]), any real solution of (1.1) will necessarily have $\lambda > 0$ because $\lambda = U/(2I)$.

The simplest known planar central configuration of the N-body problem for $N \geq 2$ is obtained by taking N equal masses at the vertices of a regular N-gon. We cannot find in the literature who was the first in knowing such planar central configurations. It is also known the existence of planar central configurations for the pn-body problem with $p = 2$ where the masses are at the vertices of two nested homothetic regular n-gons with a common center. In such configurations all the masses on the same n-gon are equal but masses on different n-gons could be different. It seems that the first in studying these nested planar central configurations was Longley [3] in 1907, later on in 1927 and 1929 Bilimovithc (see [1]) and in 1967 Klemplerer [2] also studied them. More recently they have been also studied by Moeckel and Simó in [5] and by Zhang, Xie and Zhou in [7,8]. For $p = 3$ and $p = 4$ some results on p nested regular N-gons central configurations are given by Llibre and Melo in [4].

In this paper we shall prove the following result.

Theorem 1.1 *For all $p \geq 2$ and $n \geq 2$ there exist planar central configurations of the pn-body problem where the masses are located at the vertices of p nested homothetic* *regular n-gons with a common center. In such configurations the masses on the same n-gon are equal, but masses on different n-gons could be different.*

2 Equations for the Nested n-gons Central Configurations

Given $p, n \in \mathbb{N}$ with $p, n \geq 2$, we consider pn masses at the vertices of p nested homothetic regular n-gons with a common center. In order to simplify our computations we write the positions $\mathbf{q}_k \in \mathbb{R}^2$ of the vertices of the regular n-gons as points of the complex plane. Let $m_{(\ell-1)n+k} = \mu_\ell$ and let $\mathbf{q}_{(\ell-1)n+k} = r_\ell\, e^{i\alpha_k}$ with $\alpha_k = 2\pi(k-1)/n$ for all $\ell = 1, \dots, p$ and $k = 1, \dots, n$. That is, we assume that the masses on the ℓ-th nested n-gon are equal to μ_ℓ and that the vertices of the ℓ-th nested n-gon are $\mathbf{q}_k = r_\ell\, e^{i\alpha_k}$ for all $k = 1, \dots, n$ with $r_\ell < r_{\ell+1}$ for all $\ell = 1, \dots, p-1$.

Using this notation the pn Eq. (1.1) become

$$\sum_{\substack{j=1\\ j\neq k}}^{n} \mu_1 \frac{r_1\, e^{i\alpha_k} - r_1\, e^{i\alpha_j}}{|r_1\, e^{i\alpha_k} - r_1\, e^{i\alpha_j}|^3} + \sum_{j=1}^{n} \mu_2 \frac{r_1\, e^{i\alpha_k} - r_2\, e^{i\alpha_j}}{|r_1\, e^{i\alpha_k} - r_2\, e^{i\alpha_j}|^3}$$

$$+ \cdots + \sum_{j=1}^{n} \mu_p \frac{r_1\, e^{i\alpha_k} - r_p\, e^{i\alpha_j}}{|r_1\, e^{i\alpha_k} - r_p\, e^{i\alpha_j}|^3} = \lambda\, r_1\, e^{i\alpha_k},$$

$$\sum_{j=1}^{n} \mu_1 \frac{r_2\, e^{i\alpha_k} - r_1\, e^{i\alpha_j}}{|r_2\, e^{i\alpha_k} - r_1\, e^{i\alpha_j}|^3} + \sum_{\substack{j=1\\ j\neq k}}^{n} \mu_2 \frac{r_2\, e^{i\alpha_k} - r_2\, e^{i\alpha_j}}{|r_2\, e^{i\alpha_k} - r_2\, e^{i\alpha_j}|^3}$$

$$+ \cdots + \sum_{j=1}^{n} \mu_p \frac{r_2\, e^{i\alpha_k} - r_p\, e^{i\alpha_j}}{|r_2\, e^{i\alpha_k} - r_p\, e^{i\alpha_j}|^3} = \lambda\, r_2\, e^{i\alpha_k},$$

$$\vdots$$

$$\sum_{j=1}^{n} \mu_1 \frac{r_p\, e^{i\alpha_k} - r_1\, e^{i\alpha_j}}{|r_p\, e^{i\alpha_k} - r_1\, e^{i\alpha_j}|^3} + \sum_{j=1}^{n} \mu_2 \frac{r_p\, e^{i\alpha_k} - r_2\, e^{i\alpha_j}}{|r_p\, e^{i\alpha_k} - r_2\, e^{i\alpha_j}|^3}$$

$$+ \cdots + \sum_{\substack{j=1\\ j\neq k}}^{n} \mu_p \frac{r_p\, e^{i\alpha_k} - r_p\, e^{i\alpha_j}}{|r_p\, e^{i\alpha_k} - r_p\, e^{i\alpha_j}|^3} = \lambda\, r_p\, e^{i\alpha_k},$$

for $k = 1, \dots, n$. By using the symmetry of the configuration, this system of pn equations can be reduced to an equivalent system with only p independent equations. Indeed, by dividing each equation of the system by $e^{i\alpha_k}$, we get

$$E_1 = 0, \quad E_2 = 0, \quad \dots \quad E_p = 0, \tag{2.1}$$

where

$$E_1 = \sum_{\substack{j=1 \\ j\neq k}}^{n} \frac{\mu_1}{r_1^2} \frac{1 - e^{i(\alpha_j - \alpha_k)}}{|e^{i\alpha_k} - e^{i\alpha_j}|^3} + \sum_{j=1}^{n} \mu_2 \frac{r_1 - r_2\, e^{i(\alpha_j - \alpha_k)}}{|r_1\, e^{i\alpha_k} - r_2\, e^{i\alpha_j}|^3}$$

$$+ \cdots + \sum_{j=1}^{n} \mu_p \frac{r_1 - r_p\, e^{i(\alpha_j - \alpha_k)}}{|r_1\, e^{i\alpha_k} - r_p\, e^{i\alpha_j}|^3} - \lambda\, r_1,$$

$$E_2 = \sum_{j=1}^{n} \mu_1 \frac{r_2 - r_1\, e^{i(\alpha_j - \alpha_k)}}{|r_2\, e^{i\alpha_k} - r_1\, e^{i\alpha_j}|^3} + \sum_{\substack{j=1 \\ j\neq k}}^{n} \frac{\mu_2}{r_2^2} \frac{1 - e^{i(\alpha_j - \alpha_k)}}{|e^{i\alpha_k} - e^{i\alpha_j}|^3}$$

$$+ \cdots + \sum_{j=1}^{n} \mu_p \frac{r_2 - r_p\, e^{i(\alpha_j - \alpha_k)}}{|r_2\, e^{i\alpha_k} - r_p\, e^{i\alpha_j}|^3} - \lambda\, r_2,$$

$$\vdots$$

$$E_p = \sum_{j=1}^{n} \mu_1 \frac{r_p - r_1\, e^{i(\alpha_j - \alpha_k)}}{|r_p\, e^{i\alpha_k} - r_1\, e^{i\alpha_j}|^3} + \sum_{j=1}^{n} \mu_2 \frac{r_p - r_2\, e^{i(\alpha_j - \alpha_k)}}{|r_p\, e^{i\alpha_k} - r_2\, e^{i\alpha_j}|^3}$$

$$+ \cdots + \sum_{\substack{j=1 \\ j\neq k}}^{n} \frac{\mu_p}{r_p^2} \frac{1 - e^{i(\alpha_j - \alpha_k)}}{|e^{i\alpha_k} - e^{i\alpha_j}|^3} - \lambda\, r_p,$$

for $k = 1, \ldots, n$. After some simplifications we get

$$\beta = \sum_{\substack{j=1 \\ j\neq k}}^{n} \frac{1 - e^{i(\alpha_j - \alpha_k)}}{|e^{i\alpha_k} - e^{i\alpha_j}|^3} = \frac{1}{4} \sum_{j=1}^{n-1} \csc(\pi j/n),$$

and

$$c_{m,\ell} = \sum_{j=1}^{n} \frac{r_m - r_\ell\, e^{i(\alpha_j - \alpha_k)}}{|r_m\, e^{i\alpha_k} - r_\ell\, e^{i\alpha_j}|^3} = \sum_{j=1}^{n} \frac{r_m - r_\ell \cos(2\pi j/n)}{(r_m^2 + r_\ell^2 - 2 r_m r_\ell \cos(2\pi j/n))^{3/2}},$$

where $m \neq \ell$. We note that if $m > \ell$, then $r_m > r_\ell$, and consequently $c_{m,\ell} > 0$ for all $m > \ell$. Moreover

$$\lim_{r_m \to r_{m-1}^+} c_{m,m-1} = \frac{\beta}{r_{m-1}^2} + \lim_{r_m \to r_{m-1}^+} \frac{1}{(r_m - r_{m-1})^2} = +\infty, \tag{2.2}$$

and

$$\lim_{r_m \to +\infty} c_{m,\ell} = 0, \tag{2.3}$$

for all $m > \ell$.

Without loss of generality we can take the unit of mass and length so that $\mu_1 = 1$ and $r_1 = 1$, respectively. Then system (2.1) can be written as a linear system of equations in the variables λ and μ_i for $i = 2, \ldots, p$ of the form $A\,\mathbf{x} = \mathbf{b}$ which is given by

$$\begin{pmatrix} -1 & c_{1,2} & c_{1,3} & \ldots & c_{1,p} \\ -r_2 & \beta/r_2^2 & c_{2,3} & \ldots & c_{2,p} \\ -r_3 & c_{3,2} & \beta/r_3^2 & \ldots & c_{3,p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -r_p & c_{p,2} & c_{p,3} & \ldots & \beta/r_p^2 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_p \end{pmatrix} = \begin{pmatrix} -\beta \\ -c_{2,1} \\ -c_{3,1} \\ \vdots \\ -c_{p,1} \end{pmatrix}. \tag{2.4}$$

3 Two Nested *n*-gons

For $p = 2$ it is known that for every μ_2 there exists a unique ratio r_2 for which the configuration of two nested n-gons is central, see for instance [5]. More precisely, for $p = 2$ system (2.4) becomes

$$\begin{aligned} -\lambda + c_{1,2}\,\mu_2 &= -\beta, \\ -r_2\lambda + \beta/r_2^2\,\mu_2 &= -c_{2,1}. \end{aligned} \tag{3.1}$$

Isolating λ of the first equation of system (3.1) and substituting it into the second one we get

$$\widetilde{h}(\mu_2, r_2) = (\beta/r_2^2 - r_2 c_{1,2})\mu_2 - r_2\,\beta + c_{2,1} = 0.$$

Taking $\mu = 1/\mu_2$ and $x = 1/r_2$, this equation is equivalent to equation $h(x) = 0$, where

$$h(x) = \left(x - \frac{\mu}{x^2}\right)\beta + \mu\,x\,\phi(x) + (1 + \mu\,x^2)\,\frac{d\phi}{dx},$$

and

$$\phi(x) = \sum_{j=1}^{n} \frac{1}{d_j} = \sum_{j=1}^{n} \frac{1}{(1 + x^2 - 2x\cos(2\pi j/n))^{1/2}}.$$

Notice that the expression of $h(x)$ obtained here coincides with the expression of $h(x)$ defined on page 983 in [5]. In [5] the authors proved that $\lim_{x\to 0^+} h(x) = -\infty$, $\lim_{x\to 1^-} h(x) = +\infty$, and that $h(x)$ is an increasing function of x for all $0 < x < 1$. Therefore for each $\mu > 0$ there exists a unique $0 < x_\mu < 1$ such that $h(x_\mu) = 0$. Moreover the function $x_\mu(\mu)$ is injective because $d\,x_\mu/d\,\mu > 0$ (see again [5]).

We remark that the increasing character of $h(x)$ follows from the following lemma.

Lemma 3.1 *Let* $\phi(x) = \sum_{j=1}^{n} 1/d_j$. *Then, for* $0 < x < 1$, $\phi(x)$ *and all of its derivatives are positive.*

Proof See Lemma 2 of [5].

In the initial variables μ_2 and r_2, we have that for each $\mu_2 > 0$ there exists a unique $r_2(\mu_2) > 1$ such that $\widetilde{h}(\mu_2, r_2(\mu_2)) = 0$; that is, for each $\mu_2 > 0$ there exists a unique $r_2(\mu_2) > 1$ satisfying (3.1). Since $c_{2,1} > 0$ and $\mu_2 > 0$, from the second equation of (3.1) we have that $\lambda > 0$. Therefore this solution gives a central configuration of two nested n-gons. On the other hand $r_2(\mu_2)$ is an injective function, so fixed a value of $r_2 > 1$, (3.1) has a unique solution with $\lambda > 0$ and $\mu_2 > 0$.

4 *p*-Nested *n*-gons for All $p > 2$

In order to prove Theorem 1.1 we claim that there exist $1 < r_2 < r_3 < \cdots < r_p$ such that system (2.4) has a unique solution $\lambda = \lambda(r_2, \ldots, r_p)$, $\mu_k = \mu_k(r_2, \ldots, r_p)$ for $k = 2, \ldots, p$ satisfying that $\lambda > 0$ and $\mu_k > 0$.

In Sect. 3 we have seen that the claim is true for $p = 2$. Next we prove the claim by induction. That is, we assume that the claim is true for $p - 1$ n-gons and we shall prove it for p n-gons.

Assume by induction hypothesis that $1 < \widetilde{r}_2 < \widetilde{r}_3 < \cdots < \widetilde{r}_{p-1}$ are such that system (2.4) with $p - 1$ instead of p has a unique solution $\widetilde{\lambda} = \widetilde{\lambda}(\widetilde{r}_2, \ldots, \widetilde{r}_{p-1})$, $\widetilde{\mu}_k = \widetilde{\mu}_k(\widetilde{r}_2, \ldots, \widetilde{r}_{p-1})$ for $k = 2, \ldots, p - 1$ satisfying that $\widetilde{\lambda} > 0$ and $\widetilde{\mu}_k > 0$.

We need the next result.

Lemma 4.1 *There exists* $\widetilde{r}_p > \widetilde{r}_{p-1}$ *such that* $\widetilde{\lambda} = \widetilde{\lambda}(\widetilde{r}_2, \ldots, \widetilde{r}_{p-1})$, $\widetilde{\mu}_k = \widetilde{\mu}_k(\widetilde{r}_2, \ldots, \widetilde{r}_{p-1})$ *for* $k = 2, \ldots, p - 1$ *and* $\widetilde{\mu}_p = 0$ *is a solution of* (2.4).

Proof Since $\widetilde{\mu}_p$ is taken equal to 0, we have that the first $p - 1$ Eq. of (2.4) are satisfied when $\lambda = \widetilde{\lambda}$, $\mu_k = \widetilde{\mu}_k$ for $k = 2, \ldots, p - 1$ and $\mu_p = \widetilde{\mu}_p = 0$. Moreover, substituting this solution into the last Eq. of (2.4), we get equation

$$f(r_p) = -r_p\,\widetilde{\lambda} + c_{p,2}\,\widetilde{\mu}_2 + \cdots + c_{p,p-1}\,\widetilde{\mu}_{p-1} + c_{p,1} = 0.$$

Using (2.2) and (2.3) with $m = p$ we have that

$$\lim_{r_p \to \widetilde{r}_{p-1}^{+}} f(r_p) = +\infty, \quad \text{and} \quad \lim_{r_p \to +\infty} f(r_p) = -\infty,$$

respectively. Moreover $f(r_p)$ is continuous in $r_p \in (\widetilde{r}_{p-1}, +\infty)$. Therefore there exists at least a value $r_p = \widetilde{r}_p > \widetilde{r}_{p-1}$ satisfying equation $f(r_p) = 0$. This completes the proof of Lemma 4.1. □

By using the Implicit Function Theorem we shall prove that the solution of (2.4) given in Lemma 4.1 can be continued to a solution with $\widetilde{\mu}_p > 0$.

Let $\mathbf{s} = (\lambda, \mu_2, \ldots, \mu_p, r_2, \ldots, r_p)$, we define

$$\begin{aligned} g_1(\mathbf{s}) &= -\lambda + c_{1,2}\,\mu_2 + c_{1,3}\,\mu_3 + \cdots + c_{1,p}\,\mu_p + \beta, \\ g_2(\mathbf{s}) &= -r_2\,\lambda + \beta/r_2^2\,\mu_2 + c_{2,3}\,\mu_3 + \cdots + c_{2,p}\,\mu_p + c_{2,1}, \\ &\vdots \\ g_p(\mathbf{s}) &= -r_p\,\lambda + c_{p,2}\,\mu_2 + c_{p,3}\,\mu_3 + \cdots + \beta/r_p^2\,\mu_p + c_{p,1}. \end{aligned}$$

Using this notation system (2.4) can be thought as system $g_i(\mathbf{s}) = 0$ for $i = 1, \ldots, p$. Let $\widetilde{\mathbf{s}} = (\widetilde{\lambda}, \widetilde{\mu}_2, \ldots, \widetilde{\mu}_p, \widetilde{r}_2, \ldots, \widetilde{r}_p)$ be the solution of system (2.4) given in Lemma 4.1. Applying the Implicit Function Theorem, if

$$D = \begin{vmatrix} \dfrac{\partial g_1}{\partial \lambda} & \dfrac{\partial g_1}{\partial \mu_2} & \dfrac{\partial g_1}{\partial \mu_3} & \cdots & \dfrac{\partial g_1}{\partial \mu_{p-1}} & \dfrac{\partial g_1}{\partial r_p} \\ \dfrac{\partial g_2}{\partial \lambda} & \dfrac{\partial g_2}{\partial \mu_2} & \dfrac{\partial g_2}{\partial \mu_3} & \cdots & \dfrac{\partial g_2}{\partial \mu_{p-1}} & \dfrac{\partial g_2}{\partial r_p} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \dfrac{\partial g_p}{\partial \lambda} & \dfrac{\partial g_p}{\partial \mu_2} & \dfrac{\partial g_p}{\partial \mu_3} & \cdots & \dfrac{\partial g_p}{\partial \mu_{p-1}} & \dfrac{\partial g_p}{\partial r_p} \end{vmatrix},$$

evaluated at $\mathbf{s} = \widetilde{\mathbf{s}}$ is different from zero, then for all $\mathbf{t} = (\mu_p, r_2, r_3, \ldots, r_{p-1})$ in a sufficiently small neighborhood U of $\widetilde{\mathbf{t}} = (\widetilde{\mu}_p, \widetilde{r}_2, \widetilde{r}_3, \ldots, \widetilde{r}_{p-1})$ we can find unique analytic functions $\lambda(\mathbf{t})$, $\mu_k(\mathbf{t})$ for $k = 2, \ldots, p-1$ and $r_p(\mathbf{t})$ such that they are solution of system (2.4). Of course $\lambda(\widetilde{\mathbf{t}}) = \widetilde{\lambda}$, $\mu_k(\widetilde{\mathbf{t}}) = \widetilde{\mu}_k$ for $k = 2, \ldots, p-1$ and $r_p(\widetilde{\mathbf{t}}) = \widetilde{r}_p$. Let $V = \{\mathbf{t} \in U \,:\, \mu_p > 0\}$. Since $\widetilde{\lambda} > 0$, $\widetilde{\mu}_k > 0$ for $k = 2, \ldots, p-1$ and $\widetilde{r}_p > \widetilde{r}_{p-1}$, then taking (if necessary) U more small we have that for all $\mathbf{t} \in V$, $\lambda(\mathbf{t}) > 0$, $\mu_k(\mathbf{t}) > 0$ for $k = 2, \ldots, p-1$, $r_p(\mathbf{t}) > \widetilde{r}_{p-1}$, and $\mu_p > 0$.

Next we prove that $D \neq 0$. After some computations we see that

$$D = \begin{vmatrix} -1 & c_{1,2} & c_{1,3} & \cdots & c_{1,p-1} & \dfrac{\partial c_{1,p}}{\partial r_p}\,\mu_p \\ -r_2 & \beta/r_2^2 & c_{2,3} & \cdots & c_{2,p-1} & \dfrac{\partial c_{2,p}}{\partial r_p}\,\mu_p \\ -r_3 & c_{3,2} & \beta/r_3^2 & \cdots & c_{3,p-1} & \dfrac{\partial c_{3,p}}{\partial r_p}\,\mu_p \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -r_{p-1} & c_{p-1,2} & c_{p-1,3} & \cdots & \beta/r_{p-1}^2 & \dfrac{\partial c_{p-1,p}}{\partial r_p}\,\mu_p \\ -r_p & c_{p,2} & c_{p,3} & \cdots & c_{p,p-1} & \dfrac{\partial g_p(\mathbf{s})}{\partial r_p} \end{vmatrix}_{\mathbf{s}=\widetilde{\mathbf{s}}},$$

where

$$\frac{\partial g_p(\mathbf{s})}{\partial r_p} = -\lambda + \frac{\partial c_{p,2}}{\partial r_p}\,\mu_2 + \frac{\partial c_{p,3}}{\partial r_p}\,\mu_3 + \cdots + \frac{\partial c_{p,p-1}}{\partial r_p}\,\mu_{p-1} - \frac{2\beta}{r_p^3}\,\mu_p + \frac{\partial c_{p,1}}{\partial r_p}.$$

Evaluating D at $\mu_p = 0$ we get

$$D = \left| \begin{matrix} -1 & c_{1,2} & c_{1,3} & \dots & c_{1,p-1} & 0 \\ -r_2 & \beta/r_2^2 & c_{2,3} & \dots & c_{2,p-1} & 0 \\ -r_3 & c_{3,2} & \beta/r_3^2 & \dots & c_{3,p-1} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -r_{p-1} & c_{p-1,2} & c_{p-1,3} & \dots & \beta/r_{p-1}^2 & 0 \\ -r_p & c_{p,2} & c_{p,3} & \dots & c_{p,p-1} & \dfrac{\partial g_p(\mathbf{s})}{\partial r_p} \end{matrix} \right|_{\mathbf{s}=\widetilde{\mathbf{s}}}.$$

Notice that

$$A = \left| \begin{matrix} -1 & c_{1,2} & c_{1,3} & \dots & c_{1,p-1} \\ -r_2 & \beta/r_2^2 & c_{2,3} & \dots & c_{2,p-1} \\ -r_3 & c_{3,2} & \beta/r_3^2 & \dots & c_{3,p-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -r_{p-1} & c_{p-1,2} & c_{p-1,3} & \dots & \beta/r_{p-1}^2 \end{matrix} \right|_{\mathbf{s}=\widetilde{\mathbf{s}}} \neq 0,$$

because by the hypotheses of induction $(\widetilde{\lambda}, \widetilde{\mu}_2, \dots \widetilde{\mu}_{p-1}, \widetilde{r}_2, \dots, \widetilde{r}_{p-1})$ is the unique solution of (2.4) with $p-1$ instead of p. In short $D \neq 0$ if and only if

$$\left.\frac{\partial g_p(\mathbf{s})}{\partial r_p}\right|_{\mathbf{s}=\widetilde{\mathbf{s}}} \neq 0.$$

Using the new variable $x = r_\ell/r_p$ we get

$$c_{p,\ell} = \frac{x^2}{r_\ell^2} \sum_{j=1}^{n} \frac{1 - x\,\cos(2\pi j/n)}{(x^2 + 1 - 2x\,\cos(2\pi j/n))^{3/2}} = \frac{x^2}{r_\ell^2}(\phi(x) + x\,\phi'(x)).$$

Applying Lemma 3.1 we have that

$$\frac{\partial c_{p,\ell}}{\partial x} = \frac{1}{r_\ell^2}\left(2x\,\phi(x) + 4x^2\,\phi'(x) + x^3\phi''(x)\right) > 0,$$

for $0 < x < 1$. Therefore

$$\frac{\partial c_{p,\ell}}{\partial r_p} = \frac{\partial c_{p,\ell}}{\partial x}\,\frac{\partial x}{\partial r_p} = \frac{\partial c_{p,\ell}}{\partial x}\left(-\frac{r_\ell}{r_p^2}\right) < 0,$$

for all $r_p > r_\ell$. Since $\widetilde{r}_p > \widetilde{r}_\ell$ for all $\ell = 1, \ldots, p-1$, we have that $\partial c_{p,\ell}(\widetilde{\mathbf{s}})/\partial r_p < 0$. Moreover $\widetilde{\lambda} > 0$, $\widetilde{\mu}_k > 0$ for all $k = 2, \ldots, p-1$ and $\widetilde{\mu}_p = 0$, therefore $\partial g_p(\mathbf{s})/\partial r_p < 0$ and consequently $D \neq 0$.

We have just proved that for all $\mathbf{t} = (\mu, r_2, \ldots, r_{p-1}) \in V$, system (2.4) has a solution $\lambda = \lambda(\mathbf{t})$, $\mu_k = \mu_k(\mathbf{t})$ for $k = 2, \ldots, p-1$ and $\mu_p = \mu$ satisfying that $\lambda > 0$ and $\mu_k > 0$ for all $k = 2, \ldots, p$ and such that $\lambda(\widetilde{\mathbf{t}}) = \widetilde{\lambda}$, $\mu_k(\widetilde{\mathbf{t}}) = \widetilde{\mu}_k$ for $k = 2, \ldots, p-1$ and $r_p(\widetilde{\mathbf{t}}) = \widetilde{r}_p$ with $\widetilde{\mathbf{t}} = (0, \widetilde{r}_2, \ldots, \widetilde{r}_{p-1})$. It only remains to prove that this is the unique solution of (2.4) for $1 < r_2 < \cdots < r_{p-1} < r_p(\mathbf{t})$. Obviously, if

$$C|_{\mathbf{r}=\widetilde{\mathbf{r}}} = \begin{vmatrix} -1 & c_{1,2} & \ldots & c_{1,p-1} & c_{1,p} \\ -r_2 & \beta/r_2^2 & \ldots & c_{2,p-1} & c_{2,p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -r_p & c_{p,2} & \ldots & c_{p,p-1} & \beta/r_p^2 \end{vmatrix}_{\mathbf{r}=\widetilde{\mathbf{r}}} \neq 0,$$

with $\mathbf{r} = (r_2, \ldots, r_p)$ and $\widetilde{\mathbf{r}} = (\widetilde{r}_2, \ldots, \widetilde{r}_p)$ and V is sufficiently small, then the solution of (2.4) obtained from the Implicit Function Theorem is unique. So if $C|_{\mathbf{r}=\widetilde{\mathbf{r}}} \neq 0$ then the claim is proved.

Now we prove the claim when $C|_{\mathbf{r}=\widetilde{\mathbf{r}}} = 0$. We fix the variables $(r_2, \ldots, r_{p-1}) = (\widetilde{r}_2, \ldots, \widetilde{r}_{p-1})$ and we consider C as a function of r_p. It is easy to see that, near the point $\widetilde{r}_p$, the functions $c_{p,k}$ and $c_{k,p}$ are analytic with respect to the variable r_p, so they are analytic with respect to the variable $1/r_p$. In particular,

$$c_{p,k} = \frac{n}{r_p^2} + 2\sum_{j=1}^{n} \cos(2\pi j/n)\frac{r_k}{r_p^3} + \sum_{j=1}^{n} \frac{3}{4}(1 + 3\cos(4\pi j/n))\frac{r_k^2}{r_p^4} + O\left(\frac{1}{r_p^5}\right),$$

$$c_{k,p} = -\sum_{j=1}^{n} \cos(2\pi j/n)\frac{1}{r_p^2} - \sum_{j=1}^{n} \frac{1}{2}(1 + 3\cos(4\pi j/n))\frac{r_k}{r_p^3}$$
$$-\frac{3}{8}\sum_{j=1}^{n} (3\cos(2\pi j/n) + 5\cos(6\pi j/n))\,\frac{r_k^2}{r_p^4} + O\left(\frac{1}{r_p^5}\right).$$

Since $\sum_{j=1}^{n} \cos(2\pi j\ell/n) = 0$ for all $\ell \in \mathbb{N}$ we have that

$$c_{p,k} = \frac{n}{r_p^2} + O\left(\frac{1}{r_p^4}\right),$$

$$c_{k,p} = -\frac{n}{2}\frac{r_k}{r_p^3} + O\left(\frac{1}{r_p^5}\right).$$

Then expanding C in power series of $1/r_p$ we get

$$
\begin{aligned}
C &= \frac{(-1)^p n}{2r_p^2}\begin{vmatrix} c_{1,2} & \dots & c_{1,p-1} & -1 \\ \beta/r_2^2 & \dots & c_{2,p-1} & -r_2 \\ \vdots & \vdots & \ddots & \vdots \\ c_{p-1,2} & \dots & \beta/r_{p-1}^2 & -r_{p-1} \end{vmatrix} \\
&\quad + \frac{\beta}{r_p^2}\begin{vmatrix} -1 & c_{1,2} & \dots & c_{1,p-1} \\ -r_2 & \beta/r_2^2 & \dots & c_{2,p-1} \\ \vdots & \vdots & \ddots & \vdots \\ -r_{p-1} & c_{p-1,2} & \dots & \beta/r_{p-1}^2 \end{vmatrix} + O\left(\frac{1}{r_p^4}\right) \\
&= \frac{1}{r_p^2}\left(\frac{n}{2}+\beta\right)\begin{vmatrix} -1 & c_{1,2} & \dots & c_{1,p-1} \\ -r_2 & \beta/r_2^2 & \dots & c_{2,p-1} \\ \vdots & \vdots & \ddots & \vdots \\ -r_{p-1} & c_{p-1,2} & \dots & \beta/r_{p-1}^2 \end{vmatrix} + O\left(\frac{1}{r_p^4}\right) \\
&= \frac{1}{r_p^2}\left(\frac{n}{2}+\beta\right) A + O\left(\frac{1}{r_p^4}\right).
\end{aligned}
$$

Since $A \neq 0$ and $\beta > 0$, C is not the constant function. Therefore if $C_{\mathbf{r}=\widetilde{\mathbf{r}}} = 0$ then we can find $\overline{r}_p \neq \widetilde{r}_p$ as close as we want to $\widetilde{r}_p$ such that $C_{\mathbf{r}=\overline{\mathbf{r}}} \neq 0$ where $\overline{\mathbf{r}} = (\widetilde{r}_2, \dots, \widetilde{r}_{p-1}, \overline{r}_p)$. This means that we can find $1 < \widetilde{r}_2 < \dots < \widetilde{r}_{p-1} < \overline{r}_p$ with $\overline{r}_p$ sufficiently close to $\widetilde{r}_p$ such that system (2.4) has a unique solution $\overline{\lambda}$, $\overline{\mu}_k$ for $k = 2, \dots, p$. Since $\overline{r}_p$ sufficiently close to $\widetilde{r}_p$, this solution is close to the solution $\widetilde{\lambda}$, $\widetilde{\mu}_k$ for $k = 2, \dots, p$. It only remains to prove that such solution can be found so that $\overline{\mu}_p > 0$.

If $\overline{\mu}_p = 0$, then we prove the claim by repeating the previous arguments of this section by taking $\overline{\mathbf{r}}$ instead of $\widetilde{\mathbf{r}}$. If $\overline{\mu}_p \neq 0$, then we can prove the claim by contradiction. Indeed, we have assumed that $C|_{\mathbf{r}=\widetilde{\mathbf{r}}} = 0$. This implies that the linear system (2.4) has infinitely many solutions $\lambda = \lambda(\mu)$, $\mu_k = \mu_k(\mu)$ for $k = 2, \dots, p-1$ and $\mu_p = \mu$ when $\mathbf{r} = \widetilde{\mathbf{r}}$. On the other hand, from the Implicit Function Theorem we have found unique analytic functions $\lambda = \lambda(\mu)$, $\mu_k = \mu_k(\mu)$ for $k = 2, \dots, p-1$, $\mu_p = \mu$ and $r_p(\mu)$ for μ sufficiently small satisfying system (2.4) (remember that we have assumed that $(r_2, \dots, r_{p-1}) = (\widetilde{r}_2, \dots, \widetilde{r}_{p-1})$). This means that $r_p(\mu) = \widetilde{r}_p$ for all μ sufficiently close to 0. But we have seen that for $\mu = \overline{\mu}$ system (2.4) has a unique solution with $r_p = \overline{r}_p \neq \widetilde{r}_p$. This gives the contradiction.

From here it follows the proof of Theorem 1.1 stated at the introduction.

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References

1. Hagihara, Y.: Celestial Mechanics, vol. 1. The MIT Press, Cambridge (1970)
2. Klemperer, W.B.: Some properties of rosette configurations of gravitating bodies in homographic equilibrium. Astron. J. **67**, 162–167 (1962)
3. Longley, W.R.: Some particular solutions in the problem of n bodies. Bull. Am. Math. Soc. **13**, 324–335 (1907)
4. Llibre, J., Mello, L.F.: Triple and quadruple nested central configurations for the planar n-body problem. Physica D **238**, 563–571 (2009)
5. Moeckel, R., Simó, C.: Bifurcation of spatial central configurations from planar ones. SIAM J. Math. Anal. **26**, 978–998 (1995)
6. Spivak, M.: Calculus on Manifolds: A Modern Approach to Classical Theorems of Advanced Calculus. Benjamin/Cummings, New York (1965)
7. Zhang, S., Xie, Z.: Nested regular polygon solutions of $2N$-body problem. Phys. Lett. A **281**, 149–154 (2001)
8. Zhang, S., Zhou, Q.: Periodic solutions for the $2n$-body problems. Proc. Am. Math. Soc. **131**, 2161–2170 (2002)