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Hardy's inequalities for monotone functions on partly ordered measure spaces

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We characterize the weighted Hardy inequalities for monotone functions in $\mathbb{R}^n_+$. In dimension $n = 1$, this recovers the standard theory of B_p weights. For $n > 1$, the result was previously only known for the case $p = 1$. In fact, our main theorem is proved in the more general setting of partly ordered measure spaces.

1. Introduction

The theory of weighted inequalities for the Hardy operator, acting on monotone functions in $\mathbb{R}_+$, was first introduced in [2]. Extensions of these results to higher dimensions have been considered only in very specific cases: in particular, in the diagonal case, for $p = 1$ only (see [3]). The main difficulty in this context is that the level sets of the monotone functions are not totally ordered, contrary to the one-dimensional case, where one considers intervals of the form $(0, a)$, $a > 0$. It is also worth pointing out that, with no monotonicity restriction, the boundedness of the Hardy operator is known only in dimension $n = 2$ (see [11, 13] and for an extension in the case of product weights see [4]).

In this work we characterize completely the weighted Hardy inequalities for all values of $p > 0$, namely, the boundedness of the operator

$$S : L^p_{\mathrm{dec}}(u) \to L^p(u),$$

where

$$Sf(s,t) = \frac{1}{st}\int_0^s \int_0^t f(x,y)\,\mathrm{d}y\,\mathrm{d}x,$$

and $L^p_{\mathrm{dec}}(u)$ is the cone of positive and decreasing functions, on each variable, in $L^p(u) = L^p(\mathbb{R}^2_+, u(x)\,\mathrm{d}x)$ (we consider, for simplicity, $n = 2$, although the result holds in any dimension).

The techniques we will use were introduced in [6] for the one-dimensional case, and also apply to a more general setting, which we now define.

We will consider a family of σ-finite measure spaces (X, μ_x) (where μ_x is a measure on X, for each $x \in X$), with a partial order '$\leqslant$' satisfying the following conditions.

(i) If $X_x := \{u \in X : u \leqslant x\}$, then $\leqslant$ restricted to X_x is a total order.

(ii) If D is a decreasing set, with respect to the order $\leqslant$ (i.e. χ_D is a decreasing function), then D is measurable.

(iii) $\mu_x(X_x) = 1$ (observe that X_x is a decreasing set).

(iv) If $u \in X_x$, then $\mathrm{d}\mu_x(y) = \mu_x(X_u)\,\mathrm{d}\mu_u(y)$. In particular, $\mu_x(X_u)\mu_u(X_x) = 1$.

The main examples are as follows.

(a) $X = \mathbb{R}_+$ with the usual order, and $\mu_x(E) = x^{-1}|E|$. This is the case considered in [2].

(b) X is a tree with the usual order on geodesics, and $\mu_x(E) = \mathrm{Card}(E)/|x|$, where $|x| = \mathrm{Card}([o,x])$ and $[o,x]$ is the geodesic path joining the origin o with any vertex x of the tree. (For more information on this case, see [10] and the references quoted therein.)

(c) $\mathbb{R}^2_+$ with the order given by $(a_1, b_1) \leqslant (a_2, b_2)$ if and only if $a_1 = a_2$ and $b_1 \leqslant b_2$ (we could also choose to fix the second coordinate). For $x = (a,b) \in \mathbb{R}^2_+$,

$$\mu_x(E) = b^{-1}\int_{E\cap(\{a\}\times\mathbb{R}_+)} \mathrm{d}t.$$

(d) In many cases, we can easily obtain the existence of the family of measures μ_x by taking a non-negative measure μ on X, and defining $\mu_x = \mu/\mu(X_x)$.

We now define the Hardy operator as follows:

$$Sf(x) = \int_{X_x} f(u)\,\mathrm{d}\mu_x(u).$$

This definition is similar to the one considered in [3]. For the case of $\mathbb{R}_+$,

$$Sf(x) = \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t.$$

On a tree,

$$Sf(x) = \sum_{y\in[o,x]} \frac{f(y)}{|x|}$$

and, for $\mathbb{R}^2_+$,

$$Sf(a,b) = \frac{1}{b}\int_0^b f(a,t)\,\mathrm{d}t.$$

One of the main techniques we will use is the following lemma. This is a kind of integration by parts.

Lemma 1.1. *Let $(X,\mu,\leqslant)$ be a finite measure space with a total order $\leqslant$, and $\alpha\in\mathbb{R}$. There then exists a constant, C_α, which depends only on α (and not on $(X,\mu,\leqslant)$), such that*

$$\left(\int_X \mathrm{d}\mu\right)^{\alpha} \leqslant C_\alpha \int_X \left(\int_{X_u}\mathrm{d}\mu\right)^{\alpha-1}\mathrm{d}\mu(u).$$

Proof. Since $\mu(X)<\infty$, by dividing both sides by $(\mu(X))^\alpha$ it suffices to show that

$$1\leqslant C_\alpha\int_X \varphi^{\alpha-1}(u)\,\mathrm{d}\mu(u),$$

where $\varphi(u)=\mu(X_u)$ and $\mu(X)=1$.

If $\alpha\leqslant 1$, then, using the fact that $0\leqslant\varphi\leqslant 1$, we obtain $\varphi^{\alpha-1}(u)\geqslant 1$, and

$$\int_X \varphi^{\alpha-1}(u)\,\mathrm{d}\mu(u)\geqslant \mu(X)=1.$$

If $1<\alpha\leqslant 2$, then $\varphi^{\alpha-1}(u)\geqslant\varphi(u)$ and, hence, it suffices to prove it for $\alpha=2$.

If $2<\alpha$, using Jensen's inequality,

$$\left(\int_X \varphi(u)\,\mathrm{d}\mu(u)\right)^{\alpha-1}\leqslant\int_X \varphi^{\alpha-1}(u)\,\mathrm{d}\mu(u),$$

and as before, this reduces to the case $\alpha=2$.

Finally, if $\alpha=2$,

$$\begin{aligned}\int_X\left(\int_{X_u}\mathrm{d}\mu(x)\right)\mathrm{d}\mu(u) &= \int_X\left(\int_{\{x\leqslant u\}}\mathrm{d}\mu(u)\right)\mathrm{d}\mu(x)\\ &= \int_X\left(1-\int_{\{x>u\}}\mathrm{d}\mu(u)\right)\mathrm{d}\mu(x)\\ &= 1-\int_X\int_{\{u\leqslant x\}}\mathrm{d}\mu(u)\,\mathrm{d}\mu(x)+\int_X\int_{\{u=x\}}\mathrm{d}\mu(u)\,\mathrm{d}\mu(x)\end{aligned}$$

(here we have used the fact that the order is total). Thus,

$$\int_X\left(\int_{X_u}\mathrm{d}\mu(x)\right)\mathrm{d}\mu(u)\geqslant\frac{1}{2}.$$

□

2. The weighted Hardy inequality

In this section we will prove the main theorem. In order to include all the examples, we need to consider a second, weaker order $\prec$ satisfying the following conditions.

(i) If $x \leqslant y$, then $x \prec y$.

(ii) If f is $\prec$-decreasing, then Sf is $\prec$-decreasing.

We recall that we still keep $\leqslant$ to define the operator S.

Remark 2.1. We can (and will in some cases) take $\prec$ to be $\leqslant$. In fact, we need only to check that the second condition holds for $\leqslant$: if f is $\leqslant$-decreasing, and $u \leqslant x$, then $Sf(x) \leqslant Sf(u)$ if and only if

$$\int_{X_u} f(y)[1-\mu_x(X_u)]\,\mathrm{d}\mu_u(y) - \int_{X_x \setminus X_u} f(y)\mu_x(X_u)\,\mathrm{d}\mu_u(y) \geqslant 0$$

(here we have used the fact that $\mathrm{d}\mu_x(y) = \mu_x(X_u)\,\mathrm{d}\mu_u(y)$), and this follows from the fact that $\inf f|_{X_u} \geqslant \sup f|_{X_x \setminus X_u}$.

If a function f, or a set D, is $\prec$-decreasing, it is also $\leqslant$-decreasing.

The main example we have in mind is $\leqslant$ in $\mathbb{R}^2_+$ as before, and $\prec$ the order given by the rectangles (which is clearly a weaker order): $(a_1, b_1) \prec (a_2, b_2)$, if and only if $a_1 \leqslant a_2$ and $b_1 \leqslant b_2$ (i.e. the rectangle in $\mathbb{R}^2_+$ determined by the origin and (a_1, b_1) is contained in that determined by the origin and (a_2, b_2)). To show the second condition, assume that f is a function decreasing on each variable. It is then obvious that

$$y^{-1}\int_0^y f(x,t)\,\mathrm{d}t$$

is also decreasing on each variable.

We denote by $L^p_\prec(\mathrm{d}\nu)$ the class of $\prec$-decreasing functions in $L^p(\mathrm{d}\nu)$. As a general assumption, we will only consider cases for which measurability of the functions involved always holds.

Theorem 2.2. *Let $(X, \mu_x, \leqslant)$ and $\prec$ satisfy the conditions given above, for all $x \in X$. Let $\mathrm{d}\nu$ be a measure on X, and $p > 0$. The Hardy operator then is bounded:*

$$S : L^p_\prec(\mathrm{d}\nu) \to L^p(\mathrm{d}\nu),$$

if and only if there exists a constant $C > 0$ such that, for all $\prec$-decreasing sets D,

$$\int_{X\setminus D} \mu_x^p(D \cap X_x)\,\mathrm{d}\nu(x) \leqslant C\nu(D). \tag{2.1}$$

Proof. Consider $f = \chi_D$, where D is $\prec$-decreasing. Then $(Sf)^p(x) = \mu_x^p(D \cap X_x)$, and hence

$$\begin{aligned}\|Sf\|^p_{L^p(\mathrm{d}\nu)} &= \int_X \mu_x^p(D \cap X_x)\,\mathrm{d}\nu(x)\\ &= \int_D \mu_x^p(D \cap X_x)\,\mathrm{d}\nu(x) + \int_{X\setminus D} \mu_x^p(D \cap X_x)\,\mathrm{d}\nu(x)\\ &\leqslant C\nu(D).\end{aligned}$$

Thus,

$$\int_{X\setminus D} \mu_x^p(D\cap X_x)\,\mathrm{d}\nu(x) \leqslant C\nu(D).$$

Observe that, since D is also $\leqslant$-decreasing, if $x\in D$, then $X_x\subset D$, and so $\mu_x^p(D\cap X_x)=\mu_x^p(X_x)=1$. Therefore,

$$\int_D \mu_x^p(D\cap X_x)\,\mathrm{d}\nu(x)=\nu(D).$$

Conversely, if $p\geqslant 1$ and $f\in L^p_{\prec}(\mathrm{d}\nu)$, using the lemma with $(X_x,\mu,\leqslant)$ and $\mathrm{d}\mu(u)=f(u)\,\mathrm{d}\mu_x(u)$, for a constant C that does not depend on either f or x, we have

$$\begin{aligned}(Sf)^p(x)&=\biggl(\int_{X_x} f(u)\,\mathrm{d}\mu_x(u)\biggr)^p \leqslant C\int_{X_x}\biggl(\int_{X_u} f(y)\,\mathrm{d}\mu_x(y)\biggr)^{p-1} f(u)\,\mathrm{d}\mu_x(u)\\
&=C\int_{X_x}\biggl(\int_{X_u} f(y)\,\mathrm{d}\mu_u(y)\biggr)^{p-1} f(u)\mu_x^{p-1}(X_u)\,\mathrm{d}\mu_x(u)\\
&=C\int_0^\infty\int_{\{g>t\}\cap X_x}\mu_x^{p-1}(X_u)\,\mathrm{d}\mu_x(u)\,\mathrm{d}t,\end{aligned}$$

where $g(u)=(Sf)^{p-1}(u)f(u)$. Hence,

$$\begin{aligned}\|Sf\|^p_{L^p(\mathrm{d}\nu)}&\leqslant C\int_X\int_0^\infty\int_{\{g>t\}\cap X_x}\mu_x^{p-1}(X_u)\,\mathrm{d}\mu_x(u)\,\mathrm{d}t\,\mathrm{d}\nu(x)\\
&\approx\int_X\int_0^{g(x)}\int_{\{g>t\}\cap X_x}\mu_x^{p-1}(X_u)\,\mathrm{d}\mu_x(u)\,\mathrm{d}t\,\mathrm{d}\nu(x)\\
&\qquad+\int_X\int_{g(x)}^\infty\int_{\{g>t\}\cap X_x}\mu_x^{p-1}(X_u)\,\mathrm{d}\mu_x(u)\,\mathrm{d}t\,\mathrm{d}\nu(x)\\
&=\mathrm{I}+\mathrm{II}.\end{aligned}$$

Since $X_u\subset X_x$, if $u\in X_x$, and $p\geqslant 1$, then

$$\mathrm{I}\leqslant\int_X\int_0^{g(x)}\mu_x^{p-1}(X_x)\mu_x(X_x)\,\mathrm{d}t\,\mathrm{d}\nu(x)=\int_X g(x)\,\mathrm{d}\nu(x).$$

Since both Sf and f are $\prec$-decreasing, and $p\geqslant 1$, g is $\prec$-decreasing and $\{g>t\}$ is a $\prec$-decreasing set. Also, if $u\in\{g>t\}\cap X_x$, then $X_u\subset\{g>t\}\cap X_x$, and hence

$$\int_{\{g>t\}\cap X_x}\mu_x^{p-1}(X_u)\,\mathrm{d}\mu_x(u)\leqslant\mu_x^p(\{g>t\}\cap X_x).$$

Therefore, using the hypothesis,

$$\begin{aligned}\mathrm{II}&\leqslant\int_0^\infty\int_{X\setminus\{g>t\}}\mu_x^p(\{g>t\}\cap X_x)\,\mathrm{d}\nu(x)\,\mathrm{d}t\\
&\leqslant C\int_0^\infty\int_{\{g>t\}}\mathrm{d}\nu(x)\,\mathrm{d}t=C\int_X g(x)\,\mathrm{d}\nu(x).\end{aligned}$$

Therefore, using Hölder's inequality,

$$\|Sf\|^p_{L^p(\mathrm{d}\nu)} \leqslant C \int_X (Sf)^{p-1}(x) f(x)\,\mathrm{d}\nu(x) \leqslant C\|Sf\|^{p-1}_{L^p(\mathrm{d}\nu)}\|f\|_{L^p(\mathrm{d}\nu)}.$$

From this *a priori* estimate, one may obtain the general result by a standard density argument.

If $0 < p < 1$, and $f \in L^p_{\prec}(\mathrm{d}\nu)$, set $D_t = \{f > t\}$, and

$$g_x(t) = \int_{D_t \cap X_x} \mathrm{d}\mu_x(u).$$

Then, using the embedding $L^p_{\mathrm{dec}}(t^{p-1}) \hookrightarrow L^1$ (see [14]), we have (observe that g_x is a decreasing function)

$$\begin{aligned}\biggl(\int_X \biggl(\int_{X_x} f(u)\,\mathrm{d}\mu_x(u)\biggr)^p \mathrm{d}\nu(x)\biggr)^{1/p} &= \biggl(\int_X \biggl(\int_0^\infty g_x(t)\,\mathrm{d}t\biggr)^p \mathrm{d}\nu(x)\biggr)^{1/p} \\ &\leqslant C\biggl(\int_X \int_0^\infty t^{p-1}(g_x(t))^p\,\mathrm{d}t\,\mathrm{d}\nu(x)\biggr)^{1/p}.\end{aligned}$$

Since $g_x(t) \leqslant \mu_x(X_x) = 1$, we then have

$$\int_0^\infty t^{p-1} \int_{D_t} (g_x(t))^p\,\mathrm{d}\nu(x)\,\mathrm{d}t \leqslant \int_0^\infty t^{p-1} \int_{D_t} \mathrm{d}\nu(x)\,\mathrm{d}t = \frac{1}{p}\|f\|^p_{L^p(\mathrm{d}\nu)}.$$

On the other hand, using the hypothesis (observe that $D_t \cap X_x$ is a decreasing set),

$$\int_0^\infty t^{p-1} \int_{X\setminus D_t} (g_x(t))^p\,\mathrm{d}\nu(x)\,\mathrm{d}t \leqslant C\int_0^\infty t^{p-1} \int_{D_t} \mathrm{d}\nu(x)\,\mathrm{d}t = \frac{C}{p}\|f\|^p_{L^p(\mathrm{d}\nu)}.$$

Therefore,

$$\biggl(\int_X \biggl(\int_{X_x} f(u)\,\mathrm{d}\mu_x(u)\biggr)^p \mathrm{d}\nu(x)\biggr)^{1/p} \leqslant C\|f\|_{L^p(\mathrm{d}\nu)}.$$

□

The following results follow easily by particularizing on each case condition (2.1) of theorem 2.2.

Corollary 2.3.

Case 1 (equality of orders $\prec$ and $\leqslant$). Let $(X, \mu_x, \leqslant)$ satisfy the conditions given above. Let $\mathrm{d}\nu$ be a measure on X, and $p > 0$. Then, the Hardy operator is bounded as

$$S : L^p_{\leqslant}(\mathrm{d}\nu) \to L^p(\mathrm{d}\nu)$$

if and only if there exists a constant $C > 0$ such that, for all $\leqslant$-decreasing sets D,

$$\int_{X\setminus D} \mu^p_x(D \cap X_x)\,\mathrm{d}\nu(x) \leqslant C\nu(D).$$

CASE 2 ($\mathbb{R}_+$). Condition (2.1) of theorem 2.2 is

$$\int_r^\infty \left(\frac{r}{x}\right)^p \mathrm{d}\nu(x) \leqslant C \int_0^r \mathrm{d}\nu(x),$$

for all $r > 0$, which is B_p (see [2]).

CASE 3 (tree T). Condition (2.1) of theorem 2.2 is

$$\sum_{x \in T \setminus D} \frac{|x \vee D|^p}{|x|^p} \nu(x) \leqslant C \sum_{x \in D} \nu(x),$$

where $x \vee D$ is the largest vertex in $[o, x] \cap D$.

CASE 4 ($\mathbb{R}^2_+$). Condition (2.1) of theorem 2.2 is

$$\int_{\mathbb{R}^2_+ \setminus D} \frac{|D_x|^p}{t^p} \mathrm{d}\nu(x,t) \leqslant C \int_D \mathrm{d}\nu(x,t),$$

where $D_x = \{t > 0 : (x,t) \in D\}$, and D is any decreasing set (on each variable).

REMARK 2.4. As mentioned above, the case of $\mathbb{R}_+$ was first considered in [2]. B_p weights are well understood and enjoy a very rich structure (see also [7, 8, 14] for an account of B_p and normability properties of weighted Lorentz spaces).

The discrete case $\mathbb{N}$ is a particular case of a tree, and can be found in [8]. Weights for a general tree were studied, without the monotonicity condition, in [1, 9]. It is easy to prove that a weight satisfying case 3 of corollary 2.3 must necessarily be in $B_p(\mathbb{N})$ (uniformly) on each geodesic (see [8]), but the converse is not true in general.

3. Weights in $B_p(\mathbb{R}^n_+)$

We will now show how to apply our previous result to obtain the weighted inequalities for the multidimensional Hardy operator, acting on decreasing functions. For simplicity, we will only consider the case $n = 2$, the general case being an easy extension. We first introduce the following notation:

$$\left.\begin{aligned} S_1 f(x,y) &= \frac{1}{x}\int_0^x f(s,y)\,\mathrm{d}s, \qquad S_2 f(x,y) = \frac{1}{y}\int_0^y f(x,t)\,\mathrm{d}t, \\ Sf(x,y) &= \frac{1}{xy}\int_0^x \int_0^y f(s,t)\,\mathrm{d}t\,\mathrm{d}s = S_1(S_2 f)(x,y) = S_2(S_1 f)(x,y). \end{aligned}\right\} \tag{3.1}$$

We denote by $D_x = \{t > 0 : (x,t) \in D\}$, and $D_x^y = D \cap ([0,x] \times [0,y])$. $L^p_{\mathrm{dec}}(u)$ is the usual cone of functions in $L^p(u)$, which are decreasing on each variable. We then have the following theorem.

THEOREM 3.1. *If $0 < p < \infty$, the following conditions are equivalent:*

(a) $S : L^p_{\mathrm{dec}}(u) \to L^p(u)$;

(b) *there exists a constant $C > 0$ such that, for every decreasing set D,*

$$\int_{\mathbb{R}^2_+ \setminus D} \frac{|D_x^y|^p}{(xy)^p} u(x,y)\,\mathrm{d}x\,\mathrm{d}y \leqslant C \int_D u(x,y)\,\mathrm{d}x\,\mathrm{d}y; \tag{3.2}$$

(c) $S_1, S_2 : L^p_{\mathrm{dec}}(u) \to L^p(u)$.

Proof. The fact that (a) implies (b) follows as usual: taking $f = \chi_D$, and using the fact that

$$Sf(x,y) = \frac{|D_x^y|}{xy},$$

we then see that (3.2) is a consequence of the hypothesis.

To show that (b) implies (c), we observe that if $(x,y) \notin D$, then

$$[0,x] \times D_x \subset D_x^y$$

and, hence, $x|D_x| \leqslant |D_x^y|$. Therefore,

$$\begin{aligned}\int_{\mathbb{R}^2_+ \setminus D} \frac{|D_x|^p}{y^p} u(x,y)\,\mathrm{d}x\,\mathrm{d}y &\leqslant \int_{D^c} \frac{|D_x^y|^p}{(xy)^p} u(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant C \int_D u(x,y)\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

and the result follows from case 4 of corollary 2.3. A similar result holds for S_1.

(c) implies (a): iterate and observe that $S_j f$ is decreasing if f is decreasing. □

Remark 3.2. The iteration technique used to prove theorem 3.1 can be extended very easily to other settings. For example, we could consider in $\mathbb{N}^2$ the operator

$$S(\{a_{n,m}\}_{n,m}) = \frac{1}{nm} \sum_{j=1}^{n} \sum_{k=1}^{m} a_{j,k},$$

acting on decreasing two-index sequences, and obtain the characterization of the boundedness of S on the weighted sequence spaces $\ell^p(\{u_{n,m}\}_{n,m})$, for general weights $\{u_{n,m}\}_{n,m}$, which improves some of the results in [4] proved only for product weights.

Condition (3.2) in $\mathbb{R}^n_+$ takes the form

$$\begin{aligned}\int_{\mathbb{R}^n_+ \setminus D} \frac{|D \cap ([0,x_1] \times \cdots \times [0,x_n])|^p}{(x_1 \cdots x_n)^p} u(x_1,\dots,x_n)\,\mathrm{d}x_1 \cdots \mathrm{d}x_n \\ \leqslant C \int_D u(x_1,\dots,x_n)\,\mathrm{d}x_1 \cdots \mathrm{d}x_n,\end{aligned}$$

which will be denoted by $u \in B_p(\mathbb{R}^n_+)$. Observe that, since $|D \cap ([0,x_1] \times \cdots \times [0,x_n])| \leqslant x_1 \cdots x_n$, we obtain $B_p(\mathbb{R}^n_+) \subset B_q(\mathbb{R}^n_+)$, if $p < q$.

We will now prove that, as in the one-dimensional case (see [2] for the original result and [12] for a different proof, related to the one we will use), $B_p(\mathbb{R}^n_+)$ satisfies the $p - \varepsilon$ condition.

Theorem 3.3. *If $u \in B_p(\mathbb{R}^n_+)$, $1 \leqslant p < \infty$, then there exists an $\varepsilon > 0$ such that $u \in B_{p-\varepsilon}(\mathbb{R}^n_+)$.*

Proof. We will consider the case $n = 2$ only ($n \geqslant 3$ follows similarly). Using theorem 3.1, it suffices to show that $S_j : L^{p-\varepsilon}_{\mathrm{dec}}(u) \to L^{p-\varepsilon}(u)$, $j = 1, 2$, and, by symmetry, we may consider only the case $j=2$. Take any decreasing set $D\subset\mathbb{R}^2_+$. There then exists a decreasing function $h:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$D=\{(s,t)\in\mathbb{R}^2_+;\ 0<t<h(s)\}.$$

Therefore, case 4 of corollary 2.3 gives

$$\int_0^\infty\int_{h(s)}^\infty \frac{h^p(s)}{t^p}u(s,t)\,\mathrm{d}t\,\mathrm{d}s\leqslant C\int_0^\infty\int_0^{h(s)}u(s,t)\,\mathrm{d}t\,\mathrm{d}s.$$

With $f=\chi_D$, we have

$$S_2f(s,t)=\begin{cases}1 & \text{if } 0<t\leqslant h(s),\\ \dfrac{h(s)}{t} & \text{if } 0\leqslant h(s)<t.\end{cases}$$

By iterating, we can prove that, for every $m\in\mathbb{N}$,

$$S_2^m f(s,t)=S_2\circ\overset{m}{\cdots}\circ S_2f(s,t)=\begin{cases}1 & \text{if } 0<t\leqslant h(s),\\ \dfrac{h(s)}{t}\displaystyle\sum_{j=0}^{m-1}\frac{1}{j!}\log^j\frac{t}{h(s)} & \text{if } 0\leqslant h(s)<t.\end{cases}$$

Hence, if $h(s)<t$, we find that the inequality

$$(S_2^m f(s,t))^p\geqslant\left(\frac{h(s)}{t}\right)^p\frac{1}{(m-1)!}\log^{m-1}\frac{t}{h(s)}$$

follows easily and that

$$\int_0^\infty\int_{h(s)}^\infty\left(\frac{h(s)}{t}\right)^p\frac{1}{(m-1)!}\log^{m-1}\frac{t}{h(s)}u(s,t)\,\mathrm{d}t\,\mathrm{d}s\leqslant C^m\int_0^\infty\int_0^{h(s)}u(s,t)\,\mathrm{d}t\,\mathrm{d}s.$$

Thus, taking $\sigma>\max(C,1/p)$, and summing over m gives

$$\begin{aligned}\int_0^\infty\int_{h(s)}^\infty\left(\frac{h(s)}{t}\right)^p\sum_{m=1}^\infty\frac{1}{\sigma^{m-1}(m-1)!}\log^{m-1}\frac{t}{h(s)}u(s,t)\,\mathrm{d}t\,\mathrm{d}s\\
&=\int_0^\infty\int_{h(s)}^\infty\left(\frac{h(s)}{t}\right)^{p-1/\sigma}u(s,t)\,\mathrm{d}t\,\mathrm{d}s\\
&\leqslant\sum_{m=1}^\infty\left(\frac{C}{\sigma}\right)^m\int_0^\infty\int_0^{h(s)}u(s,t)\,\mathrm{d}t\,\mathrm{d}s\\
&=C'\int_0^\infty\int_0^{h(s)}u(s,t)\,\mathrm{d}t\,\mathrm{d}s,\end{aligned}$$

and the result follows with $\varepsilon=1/\sigma$. □

It was proved in [3] that, for the case of the identity operator (i.e. when considering embeddings), one cannot, in general, replace the condition on all decreasing sets just by taking rectangles of the form $[0,a_1]\times\cdots\times[0,a_n]$, $a_j>0$. However, in the case of product weights, both conditions were equivalent (see [3, theorem 2.5]). We will now show that, in this context,

$$u(x) = \prod_{j=1}^{n} u_j(x_j) \in B_p(\mathbb{R}_+^n)$$

factorizes very nicely as $u_j \in B_p$, for all $j \in \{1, \dots, n\}$.

THEOREM 3.4. *Let*

$$u(x) = \prod_{j=1}^{n} u_j(x_j)$$

be a product weight. The following conditions are then equivalent:

(a) $u \in B_p(\mathbb{R}_+^n)$*;*

(b) *for every* $a_j > 0$, $j \in \{1, \dots, n\}$,

$$\int_{\mathbb{R}_+^n \setminus ([0,a_1]\times\cdots\times[0,a_n])} \frac{|([0,a_1]\times\cdots\times[0,a_n]) \cap ([0,x_1]\times\cdots\times[0,x_n])|^p}{(x_1\cdots x_n)^p} u(x)\,\mathrm{d}x_1\cdots\mathrm{d}x_n \leqslant C \int_{[0,a_1]\times\cdots\times[0,a_n]} u(x)\,\mathrm{d}x_1\cdots\mathrm{d}x_n;$$

(c) $u_j \in B_p$, $j \in \{1, \dots, n\}$.

Proof. As before, and by simplicity, we will work the details only for $n = 2$.

If $u \in B_p(\mathbb{R}_+^2)$, then by evaluating (3.2) for rectangles of the form $[0, a_1] \times [0, a_2]$ we get (b).

Assuming now that (b) holds, we evaluate the condition and obtain

$$\begin{aligned}
&\int_{\mathbb{R}_+^2 \setminus ([0,a_1]\times[0,a_2])} \frac{|([0,a_1]\times[0,a_2]) \cap ([0,x_1]\times[0,x_2])|^p}{(x_1x_2)^p} u_1(x_1)u_2(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&= \left(\int_0^{a_1} u_1(x_1)\,\mathrm{d}x_1\right) a_2^p \int_{a_2}^{\infty} \frac{u_2(x_2)}{x_2^p}\,\mathrm{d}x_2 + \left(\int_0^{a_2} u_2(x_2)\,\mathrm{d}x_2\right) a_1^p \int_{a_1}^{\infty} \frac{u_1(x_1)}{x_1^p}\,\mathrm{d}x_1 \\
&\qquad + (a_1a_2)^p \left(\int_{a_1}^{\infty} \frac{u_1(x_1)}{x_1^p}\,\mathrm{d}x_1\right)\left(\int_{a_2}^{\infty} \frac{u_2(x_2)}{x_2^p}\,\mathrm{d}x_2\right) \\
&\leqslant C \left(\int_0^{a_1} u_1(x_1)\,\mathrm{d}x_1\right)\left(\int_0^{a_2} u_2(x_2)\,\mathrm{d}x_2\right),
\end{aligned}$$

from which we easily deduce that, for $j = 1, 2$,

$$a_j^p \int_{a_j}^{\infty} \frac{u_j(x_j)}{x_j^p}\,\mathrm{d}x_j \leqslant C \int_0^{a_j} u_j(x_j)\,\mathrm{d}x_j,$$

and, hence, $u_j \in B_p$.

Finally, by iterating the one-dimensional Hardy operator, and using the fact that u is a product weight, we deduce that

$$S : L^p_{\mathrm{dec}}(u) \to L^p(u),$$

which is equivalent to (a). $\square$

REMARK 3.5. The equivalence between theorem 3.4(c) and the boundedness of the Hardy operator $S : L^p_{\mathrm{dec}}(u_1u_2) \to L^p(u_1u_2)$, for the range $p \geqslant 1$, was proved in [5], by using an indirect argument related to the characterization of the normability property of some multidimensional analogues of the weighted Lorentz spaces (in particular this proof did not make use of the $B_p(\mathbb{R}^n_+)$ condition). For the case $p = 1$, one can even prove a quantitative estimate of the constant in the B_1 condition, namely, if we set

$$\|u\|_{B_1(\mathbb{R}^2_+)} = \sup_{D \text{ decreasing}} \frac{\int_{\mathbb{R}^2_+} S\chi_D(s,t)u(s,t)\,\mathrm{d}s\,\mathrm{d}t}{\int_D u(s,t)\,\mathrm{d}s\,\mathrm{d}t},$$

then $\|u_1(x_1)u_2(x_2)\|_{B_1(\mathbb{R}^2_+)} = \|u_1\|_{B_1}\|u_2\|_{B_1}$.

As we pointed out in remark 3.2, similar results to theorem 3.4 may be obtained, for product weights, in more general settings (for example, in $\mathbb{N}^2$ [4]).

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References

1 N. Arcozzi, R. Rochberg and E. Sawyer. Carleson measures for analytic Besov spaces. *Rev. Mat. Iber.* **18** (2002), 443–510.

2 M. A. Ariño and B. Muckenhoupt. Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions. *Trans. Am. Math. Soc.* **320** (1990), 727–735.

3 S. Barza, L. E. Persson and J. Soria. Sharp weighted multidimensional integral inequalities for monotone functions. *Math. Nachr.* **210** (2000), 43–58.

4 S. Barza, H. Heinig and L. E. Persson. Duality theorem over the cone of monotone functions and sequences in higher dimensions. *J. Inequal. Applicat.* **7** (2002), 79–108.

5 S. Barza, A. Kamińska, L. E. Persson and J. Soria. Mixed norm and multidimensional Lorentz spaces. *Positivity.* doi:10.1007/s11117-005-0004-3. (In the press.)

6 M. J. Carro and J. Soria. Boundedness of some integral operators. *Can. J. Math.* **45** (1993), 1155–1166.

7 M. J. Carro, A. García del Amo and J. Soria. Weak-type weights and normable Lorentz spaces. *Proc. Am. Math. Soc.* **124** (1996), 849–857.

8 M. J. Carro, J. A. Raposo and J. Soria. Recent developments in the theory of Lorentz spaces and weighted inequalities. *Mem. Am. Math. Soc.* (In the press.)

9 W. D. Evans, D. J. Harris and L. Pick. Weighted Hardy and Poincaré inequalities on trees. *J. Lond. Math. Soc.* **52** (1995), 121–136.

10 J. L. Garcia-Domingo and J. Soria. Lorentz spaces for decreasing rearrangements of functions on trees. *Math. Inequal. Applicat.* **7** (2004), 471–490.

11 A. Kufner and L. E. Persson. *Weighted inequalities of Hardy type* (World Scientific, 2003).

12 C. J. Neugebauer. Weighted norm inequalities for averaging operators of monotone functions. *Publ. Mat.* **35** (1991), 429–447.

13 E. Sawyer. Weighted inequalities for the two-dimensional Hardy operator. *Studia Math.* **82** (1985), 1–16.

14 E. Sawyer. Boundedness of classical operators on classical Lorentz spaces. *Studia Math.* **96** (1990), 145–158.

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