Central configurations of three nested regular polyhedra for the spatial $3n$–body problem

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Abstract

Three regular polyhedra are called nested if they have the same number of vertices n, the same center and the positions of the vertices of the inner polyhedron $\mathbf{r}_i$, the ones of the medium polyhedron $\mathbf{R}_i$ and the ones of the outer polyhedron $\mathcal{R}_i$ satisfy the relation $\mathbf{R}_i = \rho\,\mathbf{r}_i$ and $\mathcal{R}_i = R\,\mathbf{r}_i$ for some scale factors $R > \rho > 1$ and for all $i = 1, \ldots, n$. We consider $3n$ masses located at the vertices of three nested regular polyhedra. We assume that the masses of the inner polyhedron are equal to m_1, the masses of the medium one are equal to m_2, and the masses of the outer one are equal to m_3. We prove that if the ratios of the masses m_2/m_1 and m_3/m_1 and the scale factors ρ and R satisfy two convenient relations, then this configuration is central for the $3n$–body problem. Moreover there is some numerical evidence that, first, fixed two values of the ratios m_2/m_1 and m_3/m_1, the $3n$–body problem has a unique central configuration of this type; and second that the number of nested regular polyhedra with the same number of vertices forming a central configuration for convenient masses and sizes is arbitrary.

Key words: $3n$–body problem, spatial central configurations, nested regular polyhedra
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1 Introduction

The equations of motion of the N–body problem in the ℓ–dimensional space with $\ell = 2, 3$ are

$$m_i \ddot{\mathbf{q}}_i = -\sum_{j=1,\, j\neq i}^{N} G\, m_i\, m_j \frac{\mathbf{q}_i - \mathbf{q}_j}{|\mathbf{q}_i - \mathbf{q}_j|^3} , \qquad i = 1, \ldots, N ,$$

where $\mathbf{q}_i \in \mathbb{R}^\ell$ is the position vector of the punctual mass m_i in an inertial coordinate system and G is the gravitational constant which can be taken equal to one by choosing conveniently the unit of time. We take the center of mass $\sum_{i=1}^N m_i\, \mathbf{q}_i / \sum_{i=1}^N m_i$ of the system at the origin of $\mathbb{R}^{\ell N}$. The *configuration space* of the N–body problem in $\mathbb{R}^\ell$ is defined by

$$\mathcal{E} = \{(\mathbf{q}_1, \ldots, \mathbf{q}_N) \in \mathbb{R}^{\ell N} \; : \; \sum_{i=1}^N m_i\, \mathbf{q}_i = 0, \;\; \mathbf{q}_i \neq \mathbf{q}_j, \;\; \text{for } i \neq j\} \; .$$

Given a set of masses $m_1, \ldots, m_N$, a configuration $(\mathbf{q}_1, \ldots, \mathbf{q}_N) \in \mathcal{E}$ is *central* if there exists a positive constant λ such that

$$\ddot{\mathbf{q}}_i = -\lambda\, \mathbf{q}_i \; , \qquad i = 1, \ldots, N \; . \tag{1}$$

That is if the acceleration $\ddot{\mathbf{q}}_i$ of each point mass m_i is proportional to its position $\mathbf{q}_i$ relative to the center of mass of the system and is directed towards this center.

The central configurations of the N–body problem are important because: every motion starting and ending in a total collision is asymptotic to a central configuration; they allow to compute all the homographic solutions; every parabolic motion of the N bodies (i.e. the N bodies tend to infinity as the time tends to infinity with zero radial velocity) is asymptotic to a central configuration (see [9,2]); moreover there is a relation between central configurations and the bifurcations of the hypersurfaces of constant energy and angular momentum (see [10]);...

Two central configurations in $\mathbb{R}^\ell$ are in the same *class* if there exists a rotation and a homothecy of $\mathbb{R}^\ell$ which transform one into the other. It is known that there are five classes of central configurations of the 3–body problem. Only partial results on central configurations are known for $N > 3$.

A central configuration of $\mathbb{R}^\ell$ is *planar* if the configuration of the N bodies is contained in a plane, and is *spatial* if does not exist a plane containing the configuration of the N bodies.

The simplest known planar central configuration of the N–body problem for

$N \geqslant 2$ is obtained by taking N equal masses at the vertices of a regular N–gon. If we take N equal masses at the vertices of a regular polyhedron with N vertices, then we obtain a spatial central configuration of the N–body problem (see [1]).

It is also known the existence of planar central configurations for the $2n$–body problem where the masses are at the vertices of two nested regular n–gons with a common center. For such configurations all the masses on the same n–gon are equal but masses on different n–gons could be different. It seems that the first in studying these nested planar central configurations was Longley [8] in 1907, later on in 1927 and 1929 Bilimovitch (see [4]) and in 1967 Klemplerer [5] also studied them. More recently they have been also studied in [11,12]. It is also know the existence of planar central configurations for the 9–body problem where the masses are at the vertices of three nested equilateral triangles with a common center, see [6,8].

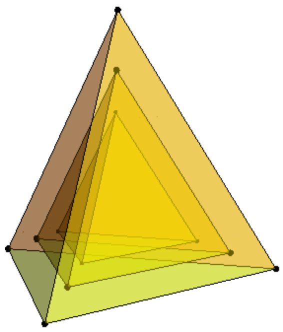
The generalization of the planar nested central configurations of the $2n$–body problem to spatial nested central configurations where the masses are located at the vertices of two nested regular polyhedra of n vertices has been studied for the tetrahedron and octahedron in [13,7], and for all types of regular polyhedra in [3].

We say that three regular polyhedra are *nested* if they have the same number of vertices n, the same center and the positions of the vertices of the inner polyhedron $\mathbf{r}_i$, the ones of the medium polyhedron $\mathbf{R}_i$ and the ones of the outer polyhedron $\mathcal{R}_i$ satisfy the relation $\mathbf{R}_i = \rho\,\mathbf{r}_i$ and $\mathcal{R}_i = R\,\mathbf{r}_i$ for some *scale factors* $R > \rho > 1$ and for all $i = 1, \ldots, n$.

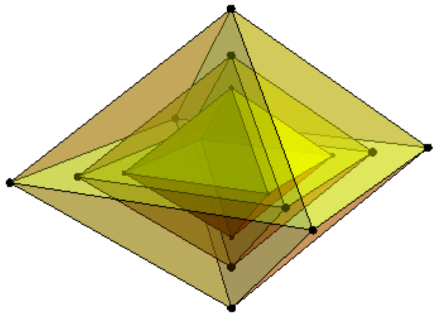
There are five regular polyhedra: the tetrahedron, the octahedron, the cube, the icosahedron and the dodecahedron with 4, 6, 8, 12 and 20 vertices, respectively. In this paper we prove that convenient masses at the vertices of three nested regular polyhedra (see Fig. 1) give spatial central configurations for the $3n$–body problem in $\mathbb{R}^3$. More precisely, we prove the following result.

Theorem 1 *We consider $3n$ masses at the vertices of three nested regular polyhedra of n vertices each one, where n can be either 4, 6, 8, 12 or 20. Assume that the masses of the inner polyhedron are equal to m_1, the masses of the medium one are equal to m_2, and the masses of the outer one are equal to m_3. If the ratios of the masses m_2/m_1 and m_3/m_1 and the scale factors ρ and R satisfy two convenient relations (i.e. relations (4)), then this configuration is central.*

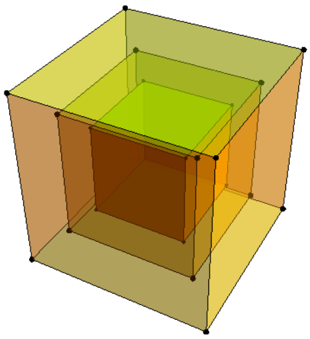
Theorem1 is the summary of the results obtained for the nested regular tetrahedra, octahedra, cube, icosahedra and dodecahedra which are given in Sections 3, 4, 5, 6, and 7, respectively.



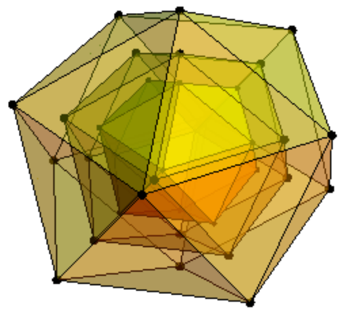
(a) Nested regular tetrahedra



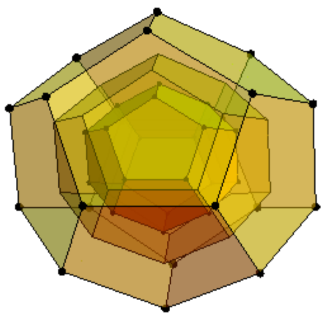
(b) Nested regular octahedra



(c) Nested regular cube



(d) Nested regular icosahedra



(e) Nested regular dodecahedra

Fig. 1. Nested regular polyhedra

There is some numerical evidence that the following two conjectures can be true.

Conjecture 2 *Fixed two values of the ratios* m_2/m_1 *and* m_3/m_1*, the* $3n$*–body problem has a unique central configuration of the type given in Theorem 1.*

Conjecture 3 *Theorem 1 can be extended to spatial central configurations of the* pn*–body problem with* p *nested regular polyhedra of* n *vertices each one for all* $p \geqslant 4$*.*

In Section 8, we find central configurations of the type given in Conjecture 3 for all regular polyhedra and for $p = 4, \ldots, 10$ when all the masses are equal. So at least in these particular cases Conjecture 3 holds. A similar conjecture for nested regular n–gons has been made in [6].

If we put an additional mass m_0 at the common center of masses of the nested polyhedra, then for all $m_0 \geqslant 0$ the resulting configuration is central when the ratios of the masses and the ratios of the length of the edges of the polyhedra satisfy some convenient relations. Notice that these relations will depend on m_0. More explicitly we shall prove in Section 9 the next result.

Theorem 4 *We consider* $3n$ *masses at the vertices of three nested regular polyhedra of* n *vertices each one, where* n *can be either 4, 6, 8, 12 or 20, and an additional mass* m_0 *at the origin. Assume that the masses of the inner polyhedron are equal to* m_1*, the masses of the medium one are equal to* m_2*, and the masses of the outer one are equal to* m_3*. If the ratios of the masses* m_2/m_1 *and* m_3/m_1 *and the scale factors* ρ *and* R *satisfy two convenient relations (i.e. relations (4)), then this configuration is central for the* $3n + 1$*–body problem.*

2 Preliminary Results

Assume that $\mathbf{q}_i = (x_i, y_i, z_i) \in \mathbb{R}^3$, then the equations of the spatial central configurations (1) can be written as

$$
\begin{aligned}
ex_i &= \sum_{j=1,\, j\neq i}^{N} \frac{m_j\,(x_i - x_j)}{((x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2)^{3/2}} - \lambda\, x_i = 0 \ , \\
ey_i &= \sum_{j=1,\, j\neq i}^{N} \frac{m_j\,(y_i - y_j)}{((x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2)^{3/2}} - \lambda\, y_i = 0 \ , \qquad (2) \\
ez_i &= \sum_{j=1,\, j\neq i}^{N} \frac{m_j\,(z_i - z_j)}{((x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2)^{3/2}} - \lambda\, z_i = 0 \ ,
\end{aligned}
$$

for $i = 1, \dots, N$. A central configuration of the N–body problem is a solution of (2) such that λ and m_j are positive for all $j = 1, \dots, N$.

As we will see in the following sections system (2) for the nested polyhedra configurations can be reduced to a system of the form $A\,X = b$, more precisely

$$\begin{pmatrix} 1 & f(\rho,1) & f(R,1) \\ \rho & -\beta/\rho^2 & f(R,\rho) \\ R & -g(R,\rho) & -\beta/R^2 \end{pmatrix} \begin{pmatrix} \lambda \\ m \\ M \end{pmatrix} = \begin{pmatrix} \beta \\ g(\rho,1) \\ g(R,1) \end{pmatrix}, \tag{3}$$

for some $f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $g : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ with $D = \{(x,y) \in \mathbb{R}^2 : x > y \geqslant 1\}$, and some constant $\beta > 0$. Now we give a technical result that will be useful in the following sections in order to prove that system (2) for the nested regular polyhedra has a solution satisfying that λ, m and M are positive.

Proposition 5 *Assume that*

(i) $f(x,y) > 0$ *and* $g(x,y) > 0$ *for all* $(x,y) \in D$,
(ii) $f(R,\rho) - \rho\, f(R,1) > 0$ *for all* $R > \rho > 1$,
(iii) $\lim_{x\to y} f(x,y) = \lim_{x\to y} g(x,y) = +\infty$ *and* $\lim_{x\to y} f(x,y)/g(x,y) = 1$,
(iv) $\beta x - g(x,1) > 0$ *for all* $x > \alpha$ *and for some* $\alpha > 1$, *and* $\beta x - g(x,1) < 0$ *for* $1 < x < \alpha$.

Then we can find two nonempty sets $\mathcal{D}_1, \mathcal{D}_2 \subset \{(R,\rho) \in \mathbb{R}^2 : R > \rho > 1\}$ *such that system* (3) *has a unique solution* $\lambda = \lambda(R,\rho)$, $m = m(R,\rho)$ *and* $M = M(R,\rho)$ *defined for all* $R > \rho > 1$, *satisfying that* $m(R,\rho) > 0$ *for all* $(R,\rho) \in \mathcal{D}_1$, *and* $M(R,\rho) > 0$ *for all* $(R,\rho) \in \mathcal{D}_2$. *Moreover the boundary of* $\mathcal{D}_1$ *is formed by the curve* $m(R,\rho) = 0$ *and the half–line* $R = \rho$ *with* $\rho \geqslant \alpha$, *and the boundary of* $\mathcal{D}_2$ *is formed by the curve* $M(R,\rho) = 0$ *and the two half–lines* $R = \rho$ *with* $1 < \rho \leqslant \alpha$ *and* $\rho = 1$ *with* $R > 1$. *Furthermore the boundaries of* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *meet at the point* $R = \rho = \alpha$. *See Fig. 2.*

We remark than in Fig. 2 and in other figures of this paper we have ρ in the abscissa axis and R in the ordinate axis.

PROOF. If $\det(A) \neq 0$ for all $R > \rho > 1$, then we can solve system (3) by Cramer and we get

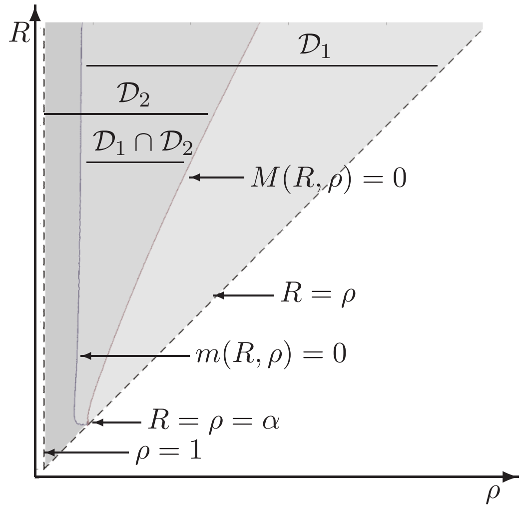


Fig. 2. The regions $\mathcal{D}_1$ and $\mathcal{D}_2$.

$$\lambda = \lambda(R,\rho) = \frac{\det(A_1)}{\det(A)} = \frac{1}{\det(A)} \begin{vmatrix} \beta & f(\rho,1) & f(R,1) \\ g(\rho,1) & -\beta/\rho^2 & f(R,\rho) \\ g(R,1) & -g(R,\rho) & -\beta/R^2 \end{vmatrix} ,$$

$$m = m(R,\rho) = \frac{\det(A_2)}{\det(A)} = \frac{1}{\det(A)} \begin{vmatrix} 1 & \beta & f(R,1) \\ \rho & g(\rho,1) & f(R,\rho) \\ R & g(R,1) & -\beta/R^2 \end{vmatrix} , \tag{4}$$

$$M = M(R,\rho) = \frac{\det(A_3)}{\det(A)} = \frac{1}{\det(A)} \begin{vmatrix} 1 & f(\rho,1) & \beta \\ \rho & -\beta/\rho^2 & g(\rho,1) \\ R & -g(R,\rho) & g(R,1) \end{vmatrix} .$$

It is easy to check that

$$\det(A) = \frac{\beta^2}{R^2\rho^2} + \frac{\beta R f(R,1)}{\rho^2} + \frac{\beta\rho f(\rho,1)}{R^2} + R f(R,\rho) f(\rho,1) + (f(R,\rho) - \rho f(R,1)) g(R,\rho) .$$

From condition (i) we have that $f(R,1) > 0$, $f(\rho,1) > 0$, $f(R,\rho) > 0$, and $g(R,\rho) > 0$ for all $R > \rho > 1$. From condition (ii) we have that $f(R,\rho) - \rho f(R,1) > 0$ for all $R > \rho > 1$. Therefore $\det(A) > 0$ for all $R > \rho > 1$ and consequently the solution $\lambda = \lambda(R,\rho)$, $m = m(R,\rho)$ and $M = M(R,\rho)$ is defined for all $R > \rho > 1$.

On the other hand we get

$$\det(A_2) = \frac{\beta^2\rho}{R^2} + f(R,\rho)(\beta R - g(R,1)) + \rho f(R,1) g(R,1) + \left(-\frac{\beta}{R^2} - R f(R,1)\right) g(\rho,1) ,$$

$$\det(A_3) = \frac{\beta^2 R}{\rho^2} - \frac{\beta g(R,1)}{\rho^2} - \beta\rho\, g(R,\rho) - \rho f(\rho,1) g(R,1) + R f(\rho,1) g(\rho,1) + g(R,\rho) g(\rho,1) .$$

Now we analyze the sign of $\det(A_2)$ and $\det(A_3)$ when we approach to the boundaries of the region $R > \rho > 1$. By using condition (iii) we can see easily that

$$\lim_{\rho\to 1}\det(A_2) = \text{sign}\left(-\frac{\beta}{R^2} - Rf(R,1)\right)\cdot\infty \ ,$$
$$\lim_{R\to \rho}\det(A_2) = \text{sign}\ (\beta R - g(R,1))\cdot\infty \ ,$$
$$\lim_{\rho\to 1}\det(A_3) = +\infty \ ,$$
$$\lim_{R\to \rho}\det(A_3) = \text{sign}\ (g(\rho,1) - \beta\rho)\cdot\infty \ .$$

Since $f(R,1) > 0$ for $R > 1$, $-\beta/R^2 - Rf(R,1) < 0$ for $R > 1$, so $\det(A_2) \to -\infty$ as $\rho \to 1$. Moreover from condition (iv) we get $\lim_{R\to\rho}\det(A_2) = -\infty$ when $1 < R < \alpha$, and $\lim_{R\to\rho}\det(A_2) = +\infty$ when $R > \alpha$. Since $\det(A_2)$ is continuous for $R > \rho > 1$, this implies that there exists a curve or a set of curves, C, with $\det(A_2) = 0$, which meets the boundary of the region $R > \rho > 1$ at the point $R = \rho = \alpha$, such that $\det(A_2) < 0$ at the left hand side of C and $\det(A_2) > 0$ at the right hand side of C. On the other hand $\lim_{\rho\to 1}\det(A_3) = +\infty$, $\lim_{R\to\rho}\det(A_3) = +\infty$ when $1 < R < \alpha$, and $\lim_{R\to\rho}\det(A_3) = -\infty$ when $R > \alpha$. Since $\det(A_3)$ is also continuous for $R > \rho > 1$, this implies that there exists a curve or a set of curves, K, with $\det(A_3) = 0$, which meets the boundary of the region $R > \rho > 1$ at the point $R = \rho = \alpha$, such that $\det(A_3) > 0$ at the left hand side of K and $\det(A_3) < 0$ at the right hand side of K. Since $\det(A) > 0$ for $R > \rho > 1$, the signs of $m(R,\rho)$ and $M(R,\rho)$ are de signs of $\det(A_2)$ and $\det(A_3)$ respectively. Therefore the sets $\mathcal{D}_1$ and $\mathcal{D}_2$ are not empty.

3 Nested tetrahedra

In this section we study the spatial central configurations of the 12–body problem when the masses are located at the vertices of three nested tetrahedra. Taking conveniently the unit of masses we can assume that all the masses of the inner tetrahedron are equal to one. We also choose the unit of length in such a way that the edges of the inner tetrahedron have length 2. Recall that the set of central configurations is invariant under homothecies.

Proposition 6 *Consider four equal masses* $m_1 = m_2 = m_3 = m_4 = 1$ *at the vertices of a regular tetrahedron with edge length* 2 *having positions* $(x_1, y_1, z_1) = (-1, -1/\sqrt{3}, -1/\sqrt{6})$, $(x_2, y_2, z_2) = (1, -1/\sqrt{3}, -1/\sqrt{6})$, $(x_3, y_3, z_3) = (0, 2/\sqrt{3}, -1/\sqrt{6})$, *and* $(x_4, y_4, z_4) = (0, 0, \sqrt{3/2})$. *Consider four additional equal masses* $m_5 = m_6 = m_7 = m_8 = m$ *at the vertices of a second nested regular tetrahedron having positions* $(x_{i+4}, y_{i+4}, z_{i+4}) = \rho\,(x_i, y_i, z_i)$ *for* $i = 1, \ldots, 4$ *with* $\rho > 1$, *and finally we consider masses* $m_9 = m_{10} = m_{11} = m_{12} = M$ *at the vertices of a third nested regular tetrahedron having positions* $(x_{i+8}, y_{i+8}, z_{i+8}) = R\,(x_i, y_i, z_i)$ *for* $i = 1, \ldots, 4$ *with* $R > \rho$ *(see*

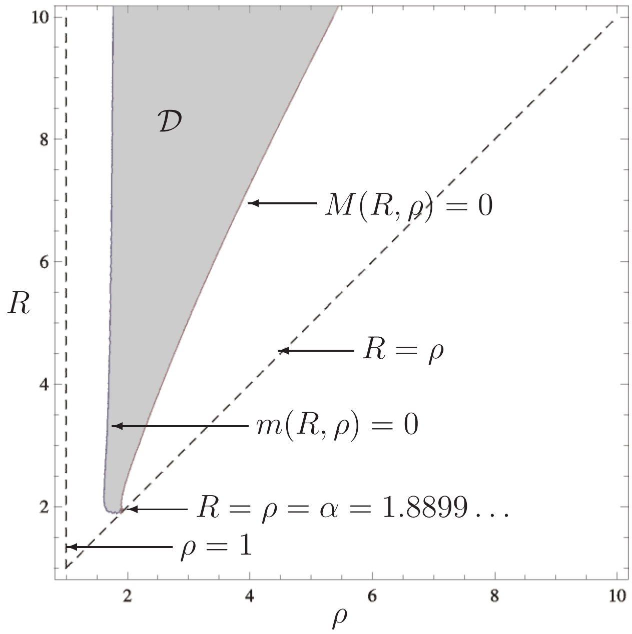


Fig. 3. The region $\mathcal{D}$.

Fig. 1(a)). Such configuration is central for the spatial 12*–body problem when* $m = m(R,\rho)$ *and* $M = M(R,\rho)$ *are given by (4) with*

$$
\begin{aligned}
f(x,y) &= \frac{2\sqrt{2/3}}{3(x-y)^2} - \frac{2\sqrt{2}(x+3y)}{(3x^2+2yx+3y^2)^{3/2}} \ , \\
g(x,y) &= \frac{2\sqrt{2/3}}{3(x-y)^2} + \frac{2\sqrt{2}(3x+y)}{(3x^2+2yx+3y^2)^{3/2}} \ ,
\end{aligned}
\tag{5}
$$

$\beta = 1/2$ *and* $(R,\rho) \in \mathcal{D} = \{(R,\rho) \in \mathbb{R}^2 \ : \ m(R,\rho) > 0, \, M(R,\rho) > 0, \, R > \rho > 1\}$ *(see Fig. 3 for the plot of* $\mathcal{D}$*).*

PROOF. It is easy to check that the positions (x_i, y_i, z_i) and the values of the masses m_i with $i = 1, \ldots, 12$ have been taken so that the center of mass of the resulting 12–body problem is located at the origin.

We substitute the positions and the values of the masses into (2). After some computations we obtain that $ex_2 = -ex_1$, $ex_6 = -ex_5$, $ex_{10} = -ex_9$, $ey_1 = ey_2 = ex_1/\sqrt{3}$, $ey_3 = -2\,ex_1/\sqrt{3}$, $ey_5 = ey_6 = ex_5/\sqrt{3}$, $ey_7 = -2\,ex_5/\sqrt{3}$, $ey_9 = ey_{10} = ex_9/\sqrt{3}$, $ey_{11} = -2\,ex_9/\sqrt{3}$, $ez_1 = ez_2 = ez_3 = ex_1/\sqrt{6}$, $ez_4 = -\sqrt{3/2}\,ex_1$, $ez_5 = ez_6 = ez_7 = ex_5/\sqrt{6}$, $ez_8 = -\sqrt{3/2}\,ex_5$, $ez_9 = ez_{10} = ez_{11} = ex_9/\sqrt{6}$, $ez_{12} = -\sqrt{3/2}\,ex_9$, and $ex_3 = ex_4 = ex_7 = ex_8 = ex_{11} = ex_{12} = ey_4 = ey_8 = ey_{12} = 0$. Therefore system (2) is equivalent to system

$$
\begin{aligned}
ex_1 = & -\frac{1}{2} + \lambda + m\left(\frac{2\sqrt{2/3}}{3(\rho-1)^2} - \frac{2\sqrt{2}(\rho+3)}{(3\rho^2+2\rho+3)^{3/2}}\right) + \\
& M\left(\frac{2\sqrt{2/3}}{3(R-1)^2} - \frac{2\sqrt{2}(R+3)}{(3R^2+2R+3)^{3/2}}\right) = 0 \ , \\
ex_5 = & -\frac{2\sqrt{2/3}}{3(\rho-1)^2} - \frac{2\sqrt{2}(3\rho+1)}{(3\rho^2+2\rho+3)^{3/2}} + \lambda\rho - \frac{m}{2\rho^2} + \\
& M\left(\frac{2\sqrt{2/3}}{3(R-\rho)^2} - \frac{2\sqrt{2}(R+3\rho)}{(3R^2+2\rho R+3\rho^2)^{3/2}}\right) = 0 \ , \\
ex_9 = & -\frac{2\sqrt{2/3}}{3(R-1)^2} - \frac{2\sqrt{2}(3R+1)}{(3R^2+2R+3)^{3/2}} + R\lambda - \\
& m\left(\frac{2\sqrt{2/3}}{3(R-\rho)^2} + \frac{2\sqrt{2}(3R+\rho)}{(3R^2+2\rho R+3\rho^2)^{3/2}}\right) - \frac{M}{2R^2} = 0 \ .
\end{aligned}
\tag{6}
$$

So system (6) can be written as system (3), with f and g given by (5) and $\beta = 1/2$. Therefore, by using Proposition 5, we prove that system (6) has a unique solution $\lambda = \lambda(R, \rho)$, $m = m(R, \rho)$ and $M = M(R, \rho)$ defined for $R > \rho > 1$. This solution gives a central configuration of the 12–body problem if and only if R and ρ are such that $\lambda(R, \rho) > 0$, $m(R, \rho) > 0$ and $M(R, \rho) > 0$.

First we prove that f and g satisfy the conditions of Proposition 5. Since equation $3x^2 + 2yx + 3y^2 = 0$ has no real solutions, f and g are well defined for all $(x, y) \in D$. From the definition of g we get $g(x, y) > 0$ for all $(x, y) \in D$. Now we see that $f(x, y) > 0$ for all $(x, y) \in D$. We consider the family of straight lines $x = a\, y$ with $a > 1$. It is clear that

$$
D = \{(x, y) \in \mathbb{R}^2 \ : \ x = a\, y, \ y \geqslant 1, \ a > 1\} \ .
$$

So in order to prove that $f(x, y) > 0$ in D it is sufficient to prove that $f(x, y)$ restricted to the half–line $x = a\, y$ with $y \geqslant 1$ is positive for all $a > 1$. It is easy to check that $f(a\, y, y) = 2\sqrt{2}/(9y^2)\, f_1(a)$ where

$$
f_1(a) = \frac{\sqrt{3}}{(a-1)^2} - \frac{9(a+3)}{(3a^2+2a+3)^{3/2}} \ .
$$

We solve equation $f_1(a) = 0$ by using the procedure described in the Appendix and we see that it has no real solutions for $a > 1$. In particular $f_1(a) > 0$ for $a > 1$. Therefore condition (i) of Proposition 5 is satisfied.

Let $F(R, \rho) = f(R, \rho) - \rho f(R, 1)$. We consider the family of straight lines $\rho = a\, R$ with $0 < a < 1$. It is clear that

$$
\{(R, \rho) \in \mathbb{R}^2 \ : \ R > \rho > 1\} = \{(R, \rho) \in \mathbb{R}^2 \ : \ \rho = a\, R, \ R > 1/a, \ 0 < a < 1\} \ .
$$

So in order to prove that $F(R,\rho) > 0$ for all $R > \rho > 1$ it is sufficient to prove that $F(R,\rho)$ restricted to the half–line $\rho = a\,R$ with $R > 1/a$ is positive for all $0 < a < 1$. Since $0 < a < 1$ and $R > 1$, the inequality $F(R, a\,R) > 0$ is equivalent to the inequality $f_1(a) - f_2(R) > 0$ where

$$f_1(a) = \frac{\sqrt{3}}{a(a-1)^2} - \frac{9(3a+1)}{a\,(3a^2+2a+3)^{3/2}},$$
$$f_2(R) = R^3\left(\frac{\sqrt{3}}{(R-1)^2} - \frac{9(R+3)}{(3R^2+2R+3)^{3/2}}\right) .$$

Now we see that f_2 is decreasing for $R > 1$. Indeed

$$f_2'(R) = \frac{\sqrt{3}(R-3)R^2}{(R-1)^3} - \frac{9R^2\,(3R^3+5R^2+21R+27)}{(3R^2+2R+3)^{5/2}} .$$

We solve equation $f_2'(R) = 0$ by using the procedure described in the Appendix and we see that it has no real solutions with $R > 1$. Since $f_2'(R) < 0$ for all $R > 1$, f_2 is decreasing in this region. On the other hand $\lim_{R\to 1/a} f_2(R) = f(a)$. So $f_2(R) < f(a)$ for all $R > 1/a$ and consequently condition (ii) of Proposition 5 is satisfied. Condition (iii) can be proved directly from the definitions of f and g.

Finally we solve the equation $\beta x - g(x,1) = 0$ (see the Appendix) and we find a unique real solution with $x > 1$ which is $x = \alpha = 1.8899915758445014\ldots$. In particular $\beta x - g(x,1) < 0$ for $1 < x < \alpha$, and $\beta x - g(x,1) > 0$ for $x > \alpha$. So condition (iv) of Proposition 5 is satisfied.

Proposition 5 assures that there exist two nonempty sets $\mathcal{D}_1, \mathcal{D}_2 \subset \{(R,\rho) \in \mathbb{R}^2\ :\ R > \rho > 1\}$ such that system (6) has a unique solution $\lambda = \lambda(R,\rho)$, $m = m(R,\rho)$ and $M = M(R,\rho)$ defined for all $R > \rho > 1$, satisfying that $m(R,\rho) > 0$ for all $(R,\rho) \in \mathcal{D}_1$, and $M(R,\rho) > 0$ for all $(R,\rho) \in \mathcal{D}_2$. Moreover $\mathcal{D}_1$ and $\mathcal{D}_2$ meet the boundary $R = \rho$ at the point $R = \rho = \alpha$. In Fig. 3 we plot the curves $m(R,\rho) = 0$ and $M(R,\rho) = 0$ and the region $\mathcal{D} = \mathcal{D}_1 \cap \mathcal{D}_2 = \{(R,\rho) \in \mathbb{R}^2\ :\ m(R,\rho) > 0,\ M(R,\rho) > 0,\ R > \rho > 1\}$. It only remains to prove that $\lambda(R,\rho) > 0$ in $\mathcal{D}$. Since $\lambda = \lambda(R,\rho)$, $m = m(R,\rho)$ and $M = M(R,\rho)$ is a solution of (6) and m and M are positive in $\mathcal{D}$, from equation $ex_9 = 0$ of (6) we get that λ is positive in $\mathcal{D}$.

In Fig. 4 we plot the level curves $m(R,\rho) = c$ and $M(R,\rho) = c$ for $c = 0, 0.1, 1, 2, 3, 5, 8, 15, 30, 100, 200$. We note that given $c_1, c_2 > 0$ Fig. 4 shows that apparently there exists a unique intersection point between the level curves $m(R,\rho) = c_1$ and $M(R,\rho) = c_2$. Therefore it seems that for each pair

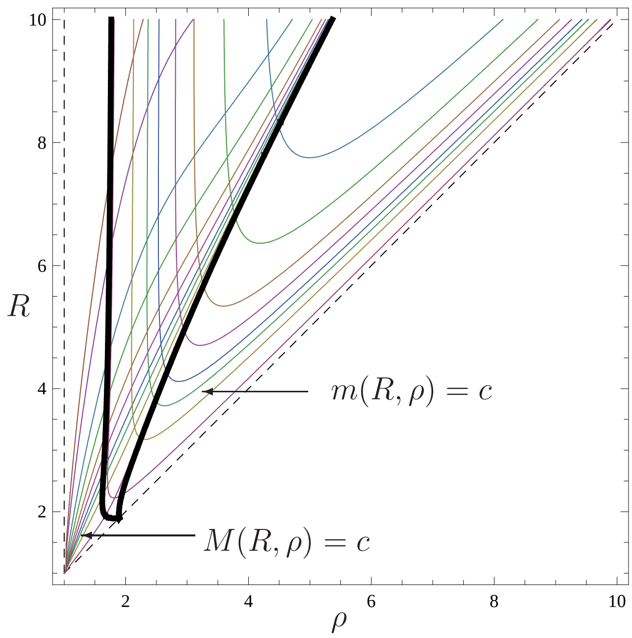


Fig. 4. The level curves $m(R,\rho) = c$ and $M(R,\rho) = c$.

of values of $m > 0$ and $M > 0$ there exists a unique $(R, \rho) \in \mathcal{D}$ such that the configuration is central. That is, for each pair of values of $m > 0$ and $M > 0$, we claim that Proposition 6 gives a unique central configuration of the spatial 12–body problem with the masses located at the vertices of three nested regular tetrahedra. This provides numerical evidence that Conjecture 2 holds for the regular tetrahedra.

In [3] it is proved that the configuration formed by four equal masses $m_1 = m_2 = m_3 = m_4 = 1$ located at the vertices of the inner tetrahedron defined in Proposition 6 and four additional equal masses $m_5 = m_6 = m_7 = m_8 = m$ located at the vertices the medium one is central when

$$m = \frac{\dfrac{(2/3)^{3/2}}{(\rho-1)^2} - \dfrac{\rho}{2} + \dfrac{2\sqrt{2}\,(3\rho+1)}{(3\rho^2+2\rho+3)^{3/2}}}{-\dfrac{1/2}{\rho^2} - \dfrac{(2/3)^{3/2}\rho}{(\rho-1)^2} + \dfrac{2\sqrt{2}\,\rho\,(\rho+3)}{(3\rho^2+2\rho+3)^{3/2}}},$$

and $\rho > \alpha = 1.8899915758445007\ldots$. Notice that m is an increasing function of ρ and $m \to +\infty$ as $\rho \to +\infty$. It is not difficult to check that this expression for m correspond to the solution of system (6) on the curve $M(R, \rho) = 0$. This means that if we take a central configuration of the 8–body problem with the masses located at the vertices of two nested regular tetrahedra with scale factor $\rho > 1$ (the inner and the medium tetrahedra defined in Proposition 6) and we put four infinitessimal masses at the vertices of a third nested regular tetrahedra with scale factor R, with $R > \rho$ and ρ and R satisfying the relation $M(R, \rho) = 0$, then the resulting configuration is central for a 12–body problem with 4 masses equal to zero. By analyzing the properties of the curve $M(R, \rho) = 0$, we see that if $\rho \to +\infty$ (or equivalently $m \to +\infty$), then $R \sim \alpha\rho$.

In a similar way if we take a central configuration of the 8–body problem with the masses located at the vertices of two nested regular tetrahedra with scale factor $R > 1$ (the inner and the outer tetrahedra defined in Proposition 6) and we put four infinitessimal masses at the vertices of a third nested regular tetrahedra with scale factor ρ, with $R > \rho$ and ρ and R satisfying the relation $m(R, \rho) = 0$, then the resulting configuration is central for another 12–body problem with 4 masses equal to zero. By analyzing the properties of the curve $m(R, \rho) = 0$, we see that $\rho \leqslant \alpha$ for all (R, ρ) satisfying $m(R, \rho) = 0$ and that if either $R \to \alpha$ (or equivalently $M \to 0$) or $R \to +\infty$ (or equivalently $M \to +\infty$), then $\rho \to \alpha$.

Notice that if we take a central configuration of the 8–body problem with the masses located at the vertices of two nested regular tetrahedra with scale factor $\rho > 1$ and we put four infinitessimal masses at the vertices of a third nested regular tetrahedra with scale factor R, with $R < 1$ (that is, the four infinitessimal masses are located at the vertices of an inner tetrahedron instead of the medium or the outer one), then we would obtain similar results for another 12–body problem with 4 masses equal to zero. In order to study this case we should change the equations because here we have chosen the unit of mass so that the masses of the inner tetrahedron are equal to one. So we do not treat this case in this work.

We note that this kind of remarks also can be considered for the other regular polyhedra but we will not do them.

4 Nested octahedra

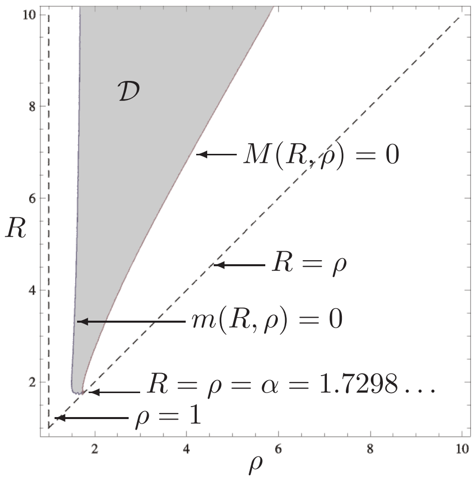
In this section we study the spatial central configurations of the 18–body problem when the masses are located at the vertices of three nested octahedra. Taking conveniently the unit of masses we can assume that all the masses of the inner tetrahedron are equal to one. We also choose the unit of length in such a way that the edges of the inner tetrahedron have length 2.

Proposition 7 *Consider six equal masses $m_i = 1$ for $i = 1, \ldots, 6$ at the vertices of a regular octahedron with edge length 2 having positions $(x_1, y_1, z_1) = (1, 0, 0)$, $(x_2, y_2, z_2) = (-1, 0, 0)$, $(x_3, y_3, z_3) = (0, 1, 0)$, $(x_4, y_4, z_4) = (0, -1, 0)$, $(x_5, y_5, z_5) = (0, 0, 1)$, $(x_6, y_6, z_6) = (0, 0, -1)$. Consider six additional equal masses $m_i = m$ for $i = 7, \ldots, 12$ at the vertices of a second nested regular octahedron having positions $(x_{i+6}, y_{i+6}, z_{i+6}) = \rho\,(x_i, y_i, z_i)$ for $i = 1, \ldots, 6$ with $\rho > 1$, and finally we consider masses $m_i = M$ for $i = 13, \ldots, 18$ at the vertices of a third nested regular octahedron having positions $(x_{i+12}, y_{i+12}, z_{i+12}) = R\,(x_i, y_i, z_i)$ for $i = 1, \ldots, 6$ with $R > \rho$ (see Fig. 1(b)). Such configuration is central for the spatial 18–body problem when $m = m(R, \rho)$ and $M = M(R, \rho)$ are given by (4) with*

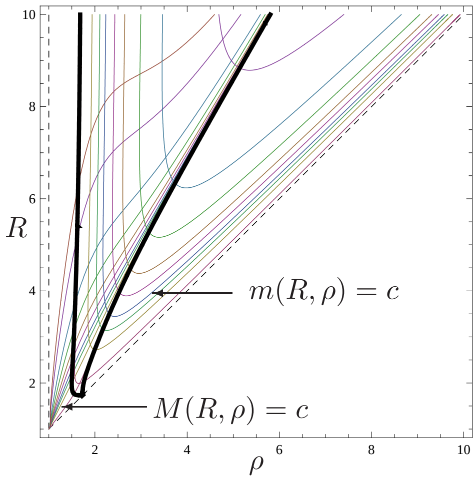
$$
\begin{aligned}
f(x,y) &= \frac{4xy}{(x^2-y^2)^2} - \frac{4y}{(x^2+y^2)^{3/2}}\ , \\
g(x,y) &= \frac{4x}{(x^2+y^2)^{3/2}} + \frac{2\,(x^2+y^2)}{(x^2-y^2)^2}\ ,
\end{aligned}
\tag{7}
$$

$\beta = \left(1 + 4\sqrt{2}\right)/4$ and $(R, \rho) \in \mathcal{D} = \{(R, \rho) \in \mathbb{R}^2 \ :\ m(R, \rho) > 0,\ M(R, \rho) > 0,\ R > \rho > 1\}$ (see Fig. 4(a) for the plot of $\mathcal{D}$).

PROOF. It is easy to check that the positions (x_i, y_i, z_i) and the values of the masses m_i with $i = 1, \ldots, 18$ have been taken so that the center of mass



(a) The region $\mathcal{D}$.



(b) The level curves $m(R,\rho)=c$ and $M(R,\rho)=c$.

Fig. 5. The functions $m(R,\rho)$ and $M(R,\rho)$.

of the resulting 18–body problem is located at the origin.

We substitute the positions and the values of the masses into (2). After some computations we obtain that $ex_3 = ex_4 = ex_5 = ex_6 = ex_9 = ex_{10} = ex_{11} = ex_{12} = ex_{15} = ex_{16} = ex_{17} = ex_{18} = ey_1 = ey_2 = ey_5 = ey_6 = ey_7 = ey_8 = ey_{11} = ey_{12} = ey_{13} = ey_{14} = ey_{17} = ey_{18} = ez_1 = ez_2 = ez_3 = ez_4 = ez_7 = ez_8 = ez_9 = ez_{10} = ez_{13} = ez_{14} = ez_{15} = ez_{16} = 0$, $ex_1 = ey_3 = ez_5 = -ex_2$, $ey_4 = ez_6 = ex_2$, $ex_7 = ey_9 = ez_{11} = -ex_8$, $ey_{10} = ez_{12} = ex_8$, $ex_{13} = ey_{15} = ez_{17} = -ex_{14}$, and $ey_{16} = ez_{18} = ex_{14}$. Therefore system (2) is equivalent to system

$$
\begin{aligned}
ex_2 = & -\frac{\left(1+4\sqrt{2}\right)}{4} + \lambda + m\left(\frac{4\rho}{\left(\rho^2-1\right)^2} - \frac{4}{\left(\rho^2+1\right)^{3/2}}\right) + \\
& M\left(\frac{4R}{\left(R^2-1\right)^2} - \frac{4}{\left(R^2+1\right)^{3/2}}\right) = 0\ , \\
ex_8 = & -\frac{2\left(\rho^2+1\right)}{\left(\rho^2-1\right)^2} - \frac{4\rho}{\left(\rho^2+1\right)^{3/2}} + \lambda\rho - \frac{\left(1+4\sqrt{2}\right)m}{4\rho^2} + \\
& M\left(\frac{4R\rho}{\left(R^2-\rho^2\right)^2} - \frac{4\rho}{\left(R^2+\rho^2\right)^{3/2}}\right) = 0\ , \\
ex_{14} = & -\frac{4R}{\left(R^2+1\right)^{3/2}} - \frac{2\left(R^2+1\right)}{\left(R^2-1\right)^2} + R\lambda - \\
& m\left(\frac{4R}{\left(R^2+\rho^2\right)^{3/2}} + \frac{2\left(R^2+\rho^2\right)}{\left(R^2-\rho^2\right)^2}\right) - \frac{\left(1+4\sqrt{2}\right)M}{4R^2} = 0\ .
\end{aligned}
\tag{8}
$$

So system (8) can be written as system (3), with f and g given by (7) and $\beta = \left(1+4\sqrt{2}\right)/4$. By using Proposition 5 we prove that system (8) has a unique solution $\lambda = \lambda(R,\rho)$, $m = m(R,\rho)$ and $M = M(R,\rho)$ defined for $R > \rho > 1$. This solution gives a central configuration of the 18–body problem if and only if R and ρ are such that $\lambda(R,\rho) > 0$, $m(R,\rho) > 0$ and $M(R,\rho) > 0$.

First we prove that f and g satisfy the conditions of Proposition 5. Since $R > \rho > 1$, f and g are well defined for all $(x,y) \in D$. It is clear from the definition of g that $g(x,y) > 0$ for all $(x,y) \in D$. As we have seen in the proof of Proposition 6, in order to prove that $f(x,y) > 0$ in D it is sufficient to prove that $f(x,y)$ restricted to the half–line $x = a\,y$ with $y \geqslant 1$ is positive for all $a > 1$. It is easy to check that $f(a\,y,y) = (4/y^2)\,f_1(a)$ where

$$
f_1(a) = \frac{a}{\left(a^2-1\right)^2} - \frac{1}{\left(a^2+1\right)^{3/2}}\ .
$$

We solve equation $f_1(a) = 0$ by using the procedure described in the Appendix and we see that it has no real solutions for $a > 1$. In particular $f_1(a) > 0$ for

$a > 1$. Therefore condition (i) of Proposition 5 is satisfied.

Let $F(R,\rho) = f(R,\rho) - \rho f(R,1)$. As we have seen in the proof of Proposition 6, in order to prove that $F(R,\rho) > 0$ for all $R > \rho > 1$ it is sufficient to prove that $F(R,\rho)$ restricted to the half–line $\rho = a\,R$ with $R > 1/a$ is positive for all $0 < a < 1$. Since $0 < a < 1$ and $R > 1$, the inequality $F(R, a\,R) > 0$ is equivalent to the inequality $f_1(a) - f_2(R) > 0$ where

$$
\begin{aligned}
f_1(a) &= \frac{1}{(a^2-1)^2} - \frac{1}{(a^2+1)^{3/2}}, \\
f_2(R) &= R^3 \left(\frac{R}{(R^2-1)^2} - \frac{1}{(R^2+1)^{3/2}} \right) .
\end{aligned}
$$

Now we see that f_2 is decreasing for $R > 1$. Indeed

$$
f_2'(R) = -\frac{4R^3}{(R^2-1)^3} - \frac{3R^2}{(R^2+1)^{5/2}} .
$$

So $f_2'(R) < 0$ for all $R > 1$ and consequently f_2 is decreasing in this region. On the other hand $\lim_{R\to 1/a} f_2(R) = f(a)$. So $f_2(R) < f(a)$ for all $R > 1/a$, and consequently condition (ii) of Proposition 5 is satisfied. Condition (iii) can be proved directly from the definitions of f and g.

Finally we solve the equation $\beta x - g(x,1) = 0$ (see the Appendix) and we find a unique real solution with $x > 1$ which is $x = \alpha = 1.7298565115043707\ldots$. In particular $\beta x - g(x,1) < 0$ for $1 < x < \alpha$, and $\beta x - g(x,1) > 0$ for $x > \alpha$. So condition (iv) of Proposition 5 is satisfied.

Proposition 5 assures that there exist two nonempty sets $\mathcal{D}_1, \mathcal{D}_2 \subset \{(R,\rho) \in \mathbb{R}^2 \,:\, R > \rho > 1\}$ such that system (8) has a unique solution $\lambda = \lambda(R,\rho)$, $m = m(R,\rho)$ and $M = M(R,\rho)$ defined for all $R > \rho > 1$, satisfying that $m(R,\rho) > 0$ for all $(R,\rho) \in \mathcal{D}_1$, and $M(R,\rho) > 0$ for all $(R,\rho) \in \mathcal{D}_2$. Moreover $\mathcal{D}_1$ and $\mathcal{D}_2$ meet the boundary $R = \rho$ at the point $R = \rho = \alpha$. In Fig. 4(a) we plot the curves $m(R,\rho) = 0$ and $M(R,\rho) = 0$ and the region $\mathcal{D} = \mathcal{D}_1 \cap \mathcal{D}_2 = \{(R,\rho) \in \mathbb{R}^2 \,:\, m(R,\rho) > 0,\, M(R,\rho) > 0,\, R > \rho > 1\}$. It only remains to prove that $\lambda(R,\rho) > 0$ in $\mathcal{D}$. Since $\lambda = \lambda(R,\rho)$, $m = m(R,\rho)$ and $M = M(R,\rho)$ is solution of (8) and m and M are positive in $\mathcal{D}$, from equation $ex_{14} = 0$ of (8) we get that λ is positive in $\mathcal{D}$.

In Fig. 4(b) we plot the level curves $m(R,\rho) = c$ and $M(R,\rho) = c$ for $c = 0, 0.1, 1, 2, 3, 5, 8, 15, 30, 100, 200$. We note that given $c_1, c_2 > 0$ Fig. 4(b) shows that apparently there exists a unique intersection point between the level curves $m(R,\rho) = c_1$ and $M(R,\rho) = c_2$. Therefore it seems that for each pair of values of $m > 0$ and $M > 0$ there exists a unique $(R, \rho) \in \mathcal{D}$ such that the configuration is central. That is, for each pair of values of $m > 0$ and $M > 0$, we claim that Proposition 7 gives a unique central configuration of the spatial 18–body problem with the masses located at the vertices of three nested regular octahedra. This provides numerical evidence that Conjecture 2 holds for the regular octahedra.

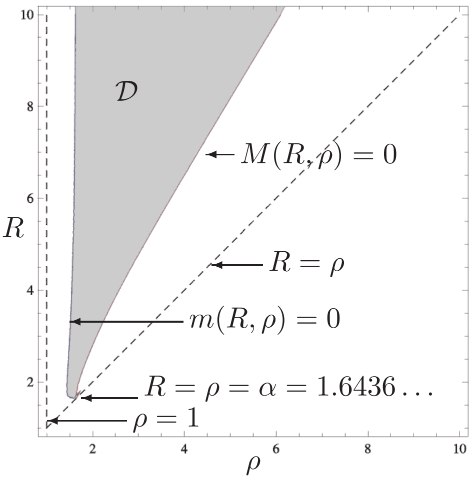
5 Nested cube

In this section we study the spatial central configurations of the 24–body problem when the masses are located at the vertices of three nested cubes. Taking conveniently the unit of masses we can assume that all the masses of the inner cube are equal to one. We also choose the unit of length in such a way that the edges of the inner cube have length 2.

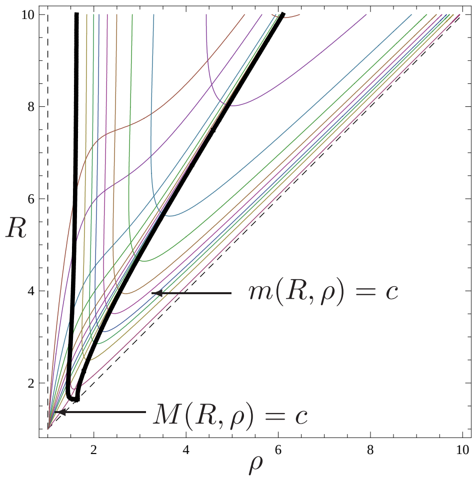
Proposition 8 *Consider eight equal masses $m_i = 1$ for $i = 1, \ldots, 8$ at the vertices of a regular cube with edge length 2 having positions $(x_1, y_1, z_1) = (1, 1, 1)$, $(x_2, y_2, z_2) = (1, 1, -1)$, $(x_3, y_3, z_3) = (1, -1, 1)$, $(x_4, y_4, z_4) = (-1, 1, 1)$, $(x_5, y_5, z_5) = (1, -1, -1)$, $(x_6, y_6, z_6) = (-1, 1, -1)$, $(x_7, y_7, z_7) = (-1, -1, 1)$, and $(x_8, y_8, z_8) = (-1, -1, -1)$. Consider eight additional equal masses $m_i = m$ for $i = 9, \ldots, 16$ at the vertices of a second nested regular cube having positions $(x_{i+8}, y_{i+8}, z_{i+8}) = \rho\,(x_i, y_i, z_i)$ for $i = 1, \ldots, 8$ with $\rho > 1$, and finally we consider masses $m_i = M$ for $i = 17, \ldots, 24$ at the vertices of a third nested regular cube having positions $(x_{i+17}, y_{i+17}, z_{i+17}) = R\,(x_i, y_i, z_i)$ for $i = 1, \ldots, 8$ with $R > \rho$ (see Fig. 1(c)). Such configuration is central for the spatial 24–body problem when $m = m(R, \rho)$ and $M = M(R, \rho)$ are given by (4) with*

$$
\begin{aligned}
f(x,y) =& \frac{x-3y}{(3x^2-2yx+3y^2)^{3/2}} - \frac{x+3y}{(3x^2+2yx+3y^2)^{3/2}} + \\
& \frac{4xy}{3\sqrt{3}\,(x^2-y^2)^2}\,, \\
g(x,y) =& \frac{3x-y}{(3x^2-2yx+3y^2)^{3/2}} + \frac{3x+y}{(3x^2+2yx+3y^2)^{3/2}} \\
& + \frac{2\,(x^2+y^2)}{3\sqrt{3}\,(x^2-y^2)^2}\,,
\end{aligned}
\tag{9}
$$

$\beta = \left(18 + 9\sqrt{2} + 2\sqrt{3}\right)/72$ and $(R, \rho) \in \mathcal{D} = \{(R, \rho) \in \mathbb{R}^2 \,:\, m(R, \rho) > 0,\ M(R, \rho) > 0,\ R > \rho > 1\}$ (see Fig. 5(a) for the plot of $\mathcal{D}$).



(a) The region $\mathcal{D}$.



(b) The level curves $m(R, \rho) = c$ and $M(R, \rho) = c$.

Fig. 6. The functions $m(R, \rho)$ and $M(R, \rho)$.

PROOF. It is easy to check that the positions (x_i, y_i, z_i) and the values of the masses m_i with $i = 1, \ldots, 24$ have been taken so that the center of mass of the resulting 24–body problem is located at the origin.

We substitute the positions and the values of the masses into (2). After some computations we obtain that $ex_1 = ex_2 = ex_3 = ex_5 = ey_1 = ey_2 = ey_4 = ey_6 = ez_1 = ez_3 = ez_4 = ez_7 = -ex_4$, $ex_6 = ex_7 = ex_8 = ey_3 = ey_5 = ey_7 = ey_8 = ez_2 = ez_5 = ez_6 = ez_8 = ex_4$, $ex_9 = ex_{10} = ex_{11} = ex_{13} = ey_9 = ey_{10} = ey_{12} = ey_{14} = ez_9 = ez_{11} = ez_{12} = ez_{15} = -ex_{12}$, $ex_{14} = ex_{15} = ex_{16} = ey_{11} = ey_{13} = ey_{15} = ey_{16} = ez_{10} = ez_{13} = ez_{14} = ez_{16} = ex_{12}$, $ex_{17} = ex_{18} = ex_{19} = ex_{21} = ey_{17} = ey_{18} = ey_{20} = ey_{22} = ez_{17} = ez_{19} = ez_{20} = ez_{23} = -ex_{20}$, and $ex_{22} = ex_{23} = ex_{24} = ey_{19} = ey_{21} = ey_{23} = ey_{24} = ez_{18} = ez_{21} = ez_{22} = ez_{24} = ex_{20}$. System (2) is equivalent to system

$$
\begin{aligned}
ex_4 &= -\beta + \lambda + m\, f(\rho, 1) + M f(R, 1) = 0 \ , \\
ex_{12} &= -g(\rho, 1) + \lambda\rho - \frac{m\beta}{\rho^2} + M f(R, \rho) = 0 \ , \\
ex_{20} &= -g(R, 1) + \lambda R - m\, g(R, \rho) - \frac{M\beta}{R^2} = 0 \ ,
\end{aligned}
\tag{10}
$$

with f and g given by (9) and $\beta = \left(18 + 9\sqrt{2} + 2\sqrt{3}\right)/72$. So system (10) can be written as system (3). By using Proposition 5 we prove that system (10) has a unique solution $\lambda = \lambda(R, \rho)$, $m = m(R, \rho)$ and $M = M(R, \rho)$ defined for $R > \rho > 1$. This solution gives a central configuration of the 24–body problem if and only if R and ρ are such that $\lambda(R, \rho) > 0$, $m(R, \rho) > 0$ and $M(R, \rho) > 0$.

First we prove that f and g satisfy the conditions of Proposition 5. Since equations $3x^2 + 2yx + 3y^2 = 0$ and $3x^2 - 2yx + 3y^2 = 0$ have no real solutions and $x > y$, f and g are well defined for all $(x, y) \in D$. It is clear from the definition of g that $g(x, y) > 0$ for all $(x, y) \in D$. As we have seen in the proof of Proposition 6, in order to prove that $f(x, y) > 0$ in D it is sufficient to prove that $f(x, y)$ restricted to the half–line $x = a\, y$ with $y \geqslant 1$ is positive for all $a > 1$. It is easy to check that $f(a\, y, y) = f_1(a)/(9y^2)$ where

$$
f_1(a) = \frac{9(a-3)}{(3a^2 - 2a + 3)^{3/2}} - \frac{9(a+3)}{(3a^2 + 2a + 3)^{3/2}} + \frac{4\sqrt{3}a}{(a^2 - 1)^2} \ .
$$

We solve equation $f_1(a) = 0$ by using the procedure described in the Appendix and we see that it has no real solutions for $a > 1$. In particular $f_1(a) > 0$ for $a > 1$. Therefore condition (i) of Proposition 5 is satisfied.

Let $F(R, \rho) = f(R, \rho) - \rho f(R, 1)$. As we have seen in the proof of Proposition 6, in order to prove that $F(R, \rho) > 0$ for all $R > \rho > 1$ it is sufficient to prove that $F(R, \rho)$ restricted to the half–line $\rho = a\, R$ with $R > 1/a$ is positive for all $0 < a < 1$. Since $0 < a < 1$ and $R > 1$, the inequality $F(R, a\,R) > 0$ is equivalent to the inequality $f_1(a) - f_2(R) > 0$ where

$$f_1(a) = \frac{4\sqrt{3}}{(a^2-1)^2} - \frac{9(3a-1)}{a\,(3a^2-2a+3)^{3/2}} - \frac{9(3a+1)}{a\,(3a^2+2a+3)^{3/2}},$$
$$f_2(R) = 9R^3\left(\frac{4R}{3\sqrt{3}\,(R^2-1)^2} + \frac{R-3}{(3R^2-2R+3)^{3/2}} - \frac{R+3}{(3R^2+2R+3)^{3/2}}\right).$$

Now we see that f_2 is decreasing for $R > 1$. Indeed

$$f_2'(R) = -\frac{16\sqrt{3}R^3}{(R^2-1)^3} + \frac{9\,(3R^3-5R^2+21R-27)\,R^2}{(3R^2-2R+3)^{5/2}} - \frac{9\,(3R^3+5R^2+21R+27)\,R^2}{(3R^2+2R+3)^{5/2}}.$$

We solve equation $f_2'(R) = 0$ by using the procedure described in the Appendix and we see that it has no real solutions with $R > 1$. Moreover $f_2'(R) < 0$ for all $R > 1$, so f_2 is decreasing in this region. On the other hand $\lim_{R\to 1/a} f_2(R) = f(a)$. So $f_2(R) < f(a)$ for all $R > 1/a$ and consequently condition (ii) of Proposition 5 is satisfied. Condition (iii) can be proved directly from the definitions of f and g.

Finally we solve the equation $\beta x - g(x,1) = 0$ (see the Appendix) and we find a unique real solution with $x > 1$ which is $x = \alpha = 1.6436467629402056\ldots$. In particular $\beta x - g(x,1) < 0$ for $1 < x < \alpha$, and $\beta x - g(x,1) > 0$ for $x > \alpha$. So condition (iv) of Proposition 5 is satisfied.

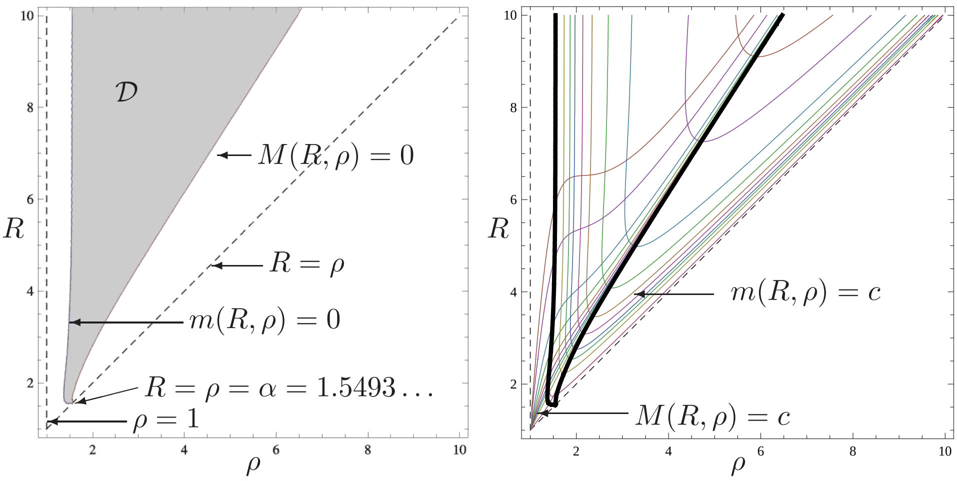
Proposition 5 assures that there exist two nonempty sets $\mathcal{D}_1, \mathcal{D}_2 \subset \{(R,\rho) \in \mathbb{R}^2 \ : \ R > \rho > 1\}$ such that system (10) has a unique solution $\lambda = \lambda(R,\rho)$, $m = m(R,\rho)$ and $M = M(R,\rho)$ defined for all $R > \rho > 1$, satisfying that $m(R,\rho) > 0$ for all $(R,\rho) \in \mathcal{D}_1$, and $M(R,\rho) > 0$ for all $(R,\rho) \in \mathcal{D}_2$. Moreover $\mathcal{D}_1$ and $\mathcal{D}_2$ meet the boundary $R = \rho$ at the point $R = \rho = \alpha$. In Fig. 5(a) we plot the curves $m(R,\rho) = 0$ and $M(R,\rho) = 0$ and the region $\mathcal{D} = \mathcal{D}_1 \cap \mathcal{D}_2 = \{(R,\rho) \in \mathbb{R}^2 \ : \ m(R,\rho) > 0, M(R,\rho) > 0, R > \rho > 1\}$. It only remains to prove that $\lambda(R,\rho) > 0$ in $\mathcal{D}$. Since $\lambda = \lambda(R,\rho)$, $m = m(R,\rho)$ and $M = M(R,\rho)$ is solution of (10) and m and M are positive in $\mathcal{D}$, from equation $ex_{20} = 0$ of (10) we get that λ is positive in $\mathcal{D}$.

In Fig. 5(b) we plot the level curves $m(R,\rho) = c$ and $M(R,\rho) = c$ for $c = 0, 0.1, 1, 2, 3, 5, 8, 15, 30, 100, 200$. We note that given $c_1, c_2 > 0$ Fig. 5(b) shows that apparently there exists a unique intersection point between the level curves $m(R,\rho) = c_1$ and $M(R,\rho) = c_2$. Therefore it seems that for each pair of values of $m > 0$ and $M > 0$ there exists a unique $(R,\rho) \in \mathcal{D}$ such that the configuration is central. That is, for each pair of values of $m > 0$ and $M > 0$, we claim that Proposition 8 gives a unique central configuration of the spatial 24–body problem with the masses located at the vertices of three nested regular cube. This provides numerical evidence that Conjecture 2 holds for the regular cube.

6 Nested icosahedra

In this section we study the spatial central configurations of the 36–body problem when the masses are located at the vertices of three nested icosahedra. Taking conveniently the unit of masses we can assume that all the masses of the inner icosahedron are equal to one. We also choose the unit of length in such a way that the edges of the inner icosahedron have length 2.

Proposition 9 *Consider twelve equal masses $m_i = 1$ for $i = 1, \ldots, 12$ at the vertices of a regular icosahedron with edge length 2 having positions $(x_1, y_1, z_1) = (0, 1, \phi)$, $(x_2, y_2, z_2) = (0, 1, -\phi)$, $(x_3, y_3, z_3) = (0, -1, \phi)$, $(x_4, y_4, z_4) = (0, -1, -\phi)$, $(x_5, y_5, z_5) = (1, \phi, 0)$, $(x_6, y_6, z_6) = (1, -\phi, 0)$, $(x_7, y_7, z_7) = (-1, \phi, 0)$, $(x_8, y_8, z_8) = (-1, -\phi, 0)$, $(x_9, y_9, z_9) = (\phi, 0, 1)$, $(x_{10}, y_{10}, z_{10}) = (\phi, 0, -1)$, $(x_{11}, y_{11}, z_{11}) = (-\phi, 0, 1)$, and $(x_{12}, y_{12}, z_{12}) = (-\phi, 0, -1)$, where $\phi = (1+\sqrt{5})/2$ is the golden ratio. Consider twelve additional equal masses $m_i = m$ for $i = 12, \ldots, 24$ at the vertices of a second nested regular icosahedron having positions $(x_{i+12}, y_{i+12}, z_{i+12}) = \rho\,(x_i, y_i, z_i)$ for $i = 1, \ldots, 12$ with $\rho > 1$, and finally we consider masses $m_i = M$ for $i = 25, \ldots, 36$ at the vertices of a third nested regular icosahedron having positions $(x_{i+25}, y_{i+25}, z_{i+25}) = R\,(x_i, y_i, z_i)$ for $i = 1, \ldots, 12$ with $R > \rho$ (see Fig. 1(d)). Such configuration is central for the spatial 36–body problem when $m = m(R,\rho)$ and $M = M(R,\rho)$ are given by (4) with*



(a) The region $\mathcal{D}$. (b) The level curves $m(R,\rho)=c$ and $M(R,\rho)=c$.

Fig. 7. The functions $m(R,\rho)$ and $M(R,\rho)$.

$$\begin{aligned} f(x,y) = & \frac{2\sqrt{2}\left(\sqrt{5}x-5y\right)}{\left(\varphi x^2-4\phi yx+\varphi y^2\right)^{3/2}} - \frac{2\sqrt{2}\left(\sqrt{5}x+5y\right)}{\left(\varphi x^2+4\phi yx+\varphi y^2\right)^{3/2}} + \\ & \frac{4\sqrt{5-2\sqrt{5}}xy}{5\left(x^2-y^2\right)^2}, \\ g(x,y) = & \frac{2\sqrt{2}\left(5x-\sqrt{5}y\right)}{\left(\varphi x^2-4\phi yx+\varphi y^2\right)^{3/2}} + \frac{2\sqrt{2}\left(5x+\sqrt{5}y\right)}{\left(\varphi x^2+4\phi yx+\varphi y^2\right)^{3/2}} + \\ & \frac{2\sqrt{5-2\sqrt{5}}\left(x^2+y^2\right)}{5\left(x^2-y^2\right)^2}, \end{aligned} \tag{11}$$

$\beta = \left(5\sqrt{5}+\sqrt{5-2\sqrt{5}}\right)/20$ *and* $(R,\rho)\in\mathcal{D}=\{(R,\rho)\in\mathbb{R}^2 \ : \ m(R,\rho)>0,\ M(R,\rho)>0,\ R>\rho>1\}$ *(see Fig. 6(a) for the plot of* $\mathcal{D}$*). Here* $\varphi = 5+\sqrt{5}$.

PROOF. It is easy to check that the positions (x_i, y_i, z_i) and the values of the masses m_i with $i = 1,\ldots,36$ have been taken so that the center of mass of the resulting 36–body problem is located at the origin.

We substitute the positions and the values of the masses into (2). After some computations we obtain that $ex_1 = ex_2 = ex_3 = ex_4 = ex_{13} = ex_{14} = ex_{15} = ex_{16} = ex_{25} = ex_{26} = ex_{27} = ex_{28} = ey_9 = ey_{10} = ey_{11} = ey_{12} = ey_{21} = ey_{22} = ey_{23} = ey_{24} = ey_{33} = ey_{34} = ey_{35} = ey_{36} = ez_5 = ez_6 = ez_7 = ez_8 = ez_{17} = ez_{18} = ez_{19} = ez_{20} = ez_{29} = ez_{30} = ez_{31} = ez_{32} = 0$, $ex_5 = ex_6 = ey_1 = ey_2 = ez_9 = ez_{11} = -ex_7$, $ex_8 = ey_3 = ey_4 = ez_{10} = ez_{12} = ex_7$, $ex_9 = ex_{10} = ey_5 = ey_7 = ez_1 = ez_3 = -ex_7\,\phi$, $ex_{11} = ex_{12} = ey_6 = ey_8 = ez_2 = ez_4 = ex_7\,\phi$, $ex_{17} = ex_{18} = ey_{13} = ey_{14} = ez_{21} = ez_{23} = -ex_{19}$, $ex_{20} = ey_{15} = ey_{16} = ez_{22} = ez_{24} = ex_{19}$, $ex_{21} = ex_{22} = ey_{17} = ey_{19} =$

$ez_{13} = ez_{15} = -ex_{19}\,\phi$, $ex_{23} = ex_{24} = ey_{18} = ey_{20} = ez_{14} = ez_{16} = ex_{19}\,\phi$, $ex_{29} = ex_{30} = ey_{25} = ey_{26} = ez_{33} = ez_{35} = -ex_{31}$, $ex_{32} = ey_{27} = ey_{28} = ez_{34} = ez_{36} = ex_{31}$, $ex_{33} = ex_{34} = ey_{29} = ey_{31} = ez_{25} = ez_{27} = -ex_{31}\,\phi$, and $ex_{35} = ex_{36} = ey_{30} = ey_{32} = ez_{26} = ez_{28} = ex_{31}\,\phi$. System (2) is equivalent to system

$$
\begin{aligned}
ex_7 &= -\beta + \lambda + m\,f(\rho,1) + Mf(R,1) = 0\ ,\\
ex_{19} &= -g(\rho,1) + \lambda\rho - \frac{m\beta}{\rho^2} + Mf(R,\rho) = 0\ ,\\
ex_{31} &= -g(R,1) + \lambda R - m\,g(R,\rho) - \frac{M\beta}{R^2} = 0\ ,
\end{aligned}
\tag{12}
$$

with f and g given by (11) and $\beta = \left(5\sqrt{5} + \sqrt{5 - 2\sqrt{5}}\right)/20$. So system (12) can be written as system (3). By using Proposition 5 we prove that system (12) has a unique solution $\lambda = \lambda(R,\rho)$, $m = m(R,\rho)$ and $M = M(R,\rho)$ defined for $R > \rho > 1$. This solution gives a central configuration of the 36–body problem if and only if R and ρ are such that $\lambda(R,\rho) > 0$, $m(R,\rho) > 0$ and $M(R,\rho) > 0$.

First we prove that f and g satisfy the conditions of Proposition 5. Since equations $\varphi x^2 - 4\phi yx + \varphi y^2 = 0$ and $\varphi x^2 + 4\phi yx + \varphi y^2 = 0$ have no real solutions and $x > y$, f and g are well defined for all $(x,y) \in D$. It is clear from the definition of g that $g(x,y) > 0$ for all $(x,y) \in D$. As we have seen in the proof of Proposition 6, in order to prove that $f(x,y) > 0$ in D it is sufficient to prove that $f(x,y)$ restricted to the half–line $x = a\,y$ with $y \geqslant 1$ is positive for all $a > 1$. It is easy to check that $f(a\,y,y) = f_1(a)/y^2$ where

$$
f_1(a) = \frac{4\sqrt{5 - 2\sqrt{5}}a}{5\left(a^2-1\right)^2} + \frac{2\sqrt{2}\left(\sqrt{5}a - 5\right)}{\left(\varphi a^2 - 4\phi a + \varphi\right)^{3/2}} - \frac{2\sqrt{2}\left(\sqrt{5}a + 5\right)}{\left(\varphi a^2 + 4\phi a + \varphi\right)^{3/2}}\ .
$$

We solve equation $f_1(a) = 0$ by using the procedure described in the Appendix and we see that it has no real solutions for $a > 1$. In particular $f_1(a) > 0$ for $a > 1$. Therefore condition (i) of Proposition 5 is satisfied.

Let $F(R,\rho) = f(R,\rho) - \rho f(R,1)$. As we have seen in the proof of Proposition 6, in order to prove that $F(R,\rho) > 0$ for all $R > \rho > 1$ it is sufficient to prove that $F(R,\rho)$ restricted to the half–line $\rho = a\,R$ with $R > 1/a$ is positive for all $0 < a < 1$. Since $0 < a < 1$ and $R > 1$, the inequality $F(R,a\,R) > 0$ is equivalent to the inequality $f_1(a) - f_2(R) > 0$ where

$$f_1(a) = \frac{4\sqrt{5-2\sqrt{5}}}{5\left(a^2-1\right)^2} - \frac{2\sqrt{2}\left(5a-\sqrt{5}\right)}{a\left(\varphi a^2-4\phi a+\varphi\right)^{3/2}} - \frac{2\sqrt{2}\left(5a+\sqrt{5}\right)}{a\left(\varphi a^2+4\phi a+\varphi\right)^{3/2}},$$

$$f_2(R) = R^3\left(\frac{4\sqrt{5-2\sqrt{5}}R}{5\left(R^2-1\right)^2} + \frac{2\sqrt{2}\left(\sqrt{5}R-5\right)}{\left(\varphi R^2-4\phi R+\varphi\right)^{3/2}} - \frac{2\sqrt{2}\left(\sqrt{5}R+5\right)}{\left(\varphi R^2+4\phi R+\varphi\right)^{3/2}}\right).$$

Now we see that f_2 is decreasing for $R > 1$. Indeed

$$f_2'(R) = \frac{10\sqrt{2}\left(2\phi R^3-\varphi R^2+14\phi R-3\varphi\right)R^2}{\left(\varphi R^2-4\phi R+\varphi\right)^{5/2}} - \frac{10\sqrt{2}\left(2\phi R^3+\varphi R^2+14\phi R+3\varphi\right)R^2}{\left(\varphi R^2+4\phi R+\varphi\right)^{5/2}} - \frac{16\sqrt{5-2\sqrt{5}}R^3}{5\left(R^2-1\right)^3}.$$

We solve equation $f_2'(R) = 0$ by using the procedure described in the Appendix and we see that it has no real solutions with $R > 1$. Moreover $f_2'(R) < 0$ for all $R > 1$, so f_2 is decreasing in this region. On the other hand $\lim_{R\to 1/a} f_2(R) = f(a)$. So $f_2(R) < f(a)$ for all $R > 1/a$ and consequently condition (ii) of Proposition 5 is satisfied. Condition (iii) can be proved directly from the definitions of f and g.

Finally we solve the equation $\beta x - g(x,1) = 0$ (see the Appendix) and we find a unique real solution with $x > 1$ which is $x = \alpha = 1.549351115673\ldots$. In particular $\beta x - g(x,1) < 0$ for $1 < x < \alpha$, and $\beta x - g(x,1) > 0$ for $x > \alpha$. So condition (iv) of Proposition 5 is satisfied.

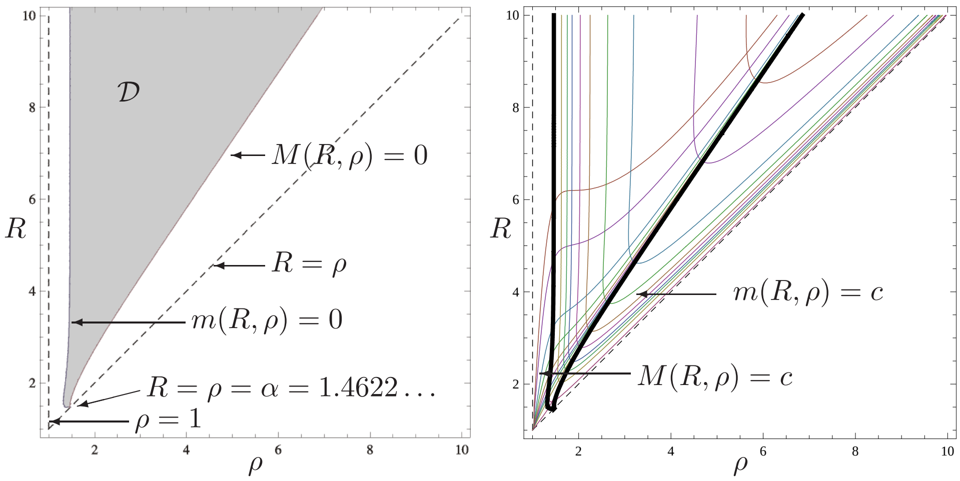
Proposition 5 assures that there exist two nonempty sets $\mathcal{D}_1, \mathcal{D}_2 \subset \{(R,\rho) \in \mathbb{R}^2 \; : \; R > \rho > 1\}$ such that system (12) has a unique solution $\lambda = \lambda(R,\rho)$, $m = m(R,\rho)$ and $M = M(R,\rho)$ defined for all $R > \rho > 1$, satisfying that $m(R,\rho) > 0$ for all $(R,\rho) \in \mathcal{D}_1$, and $M(R,\rho) > 0$ for all $(R,\rho) \in \mathcal{D}_2$. Moreover $\mathcal{D}_1$ and $\mathcal{D}_2$ meet the boundary $R = \rho$ at the point $R = \rho = \alpha$. In Fig. 6(a) we plot the curves $m(R,\rho) = 0$ and $M(R,\rho) = 0$ and the region $\mathcal{D} = \mathcal{D}_1 \cap \mathcal{D}_2 = \{(R,\rho) \in \mathbb{R}^2 \; : \; m(R,\rho) > 0, M(R,\rho) > 0, R > \rho > 1\}$. It only remains to prove that $\lambda(R,\rho) > 0$ in $\mathcal{D}$. Since $\lambda = \lambda(R,\rho)$, $m = m(R,\rho)$ and $M = M(R,\rho)$ is solution of (12) and m and M are positive in $\mathcal{D}$, from equation $ex_{31} = 0$ of (12) we get that λ is positive in $\mathcal{D}$.

In Fig. 6(b) we plot the level curves $m(R,\rho) = c$ and $M(R,\rho) = c$ for $c = 0, 0.1, 1, 2, 3, 5, 8, 15, 30, 100, 200$. We note that given $c_1, c_2 > 0$ Fig. 6(b) shows that apparently there exists a unique intersection point between the level curves $m(R,\rho)=c_1$ and $M(R,\rho)=c_2$. Therefore it seems that for each pair of values of $m>0$ and $M>0$ there exists a unique $(R,\rho)\in\mathcal{D}$ such that the configuration is central. That is, for each pair of values of $m>0$ and $M>0$, we claim that Proposition 9 gives a unique central configuration of the spatial 36–body problem with the masses located at the vertices of three nested regular icosahedra. This provides numerical evidence that Conjecture 2 holds for the regular icosahedra.

7 Nested dodecahedra

In this section we study the spatial central configurations of the 60–body problem when the masses are located at the vertices of three nested dodecahedra. Taking conveniently the unit of masses we can assume that all the masses of the inner dodecahedron are equal to one. We also choose the unit of length in such a way that the edges of the inner dodecahedron have length 2.

Proposition 10 *Consider twenty equal masses $m_i=1$ for $i=1,\dots,20$ at the vertices of a regular dodecahedron with edge length 2 having positions $(x_1,y_1,z_1)=(1,1,1)$, $(x_2,y_2,z_2)=(-1,1,1)$, $(x_3,y_3,z_3)=(1,-1,1)$, $(x_4,y_4,z_4)=(1,1,-1)$, $(x_5,y_5,z_5)=(-1,-1,1)$, $(x_6,y_6,z_6)=(-1,1,-1)$, $(x_7,y_7,z_7)=(1,-1,-1)$, $(x_8,y_8,z_8)=(-1,-1,-1)$, $(x_9,y_9,z_9)=(0,1/\phi,\phi)$, $(x_{10},y_{10},z_{10})=(0,-1/\phi,\phi)$, $(x_{11},y_{11},z_{11})=(0,1/\phi,-\phi)$, $(x_{12},y_{12},z_{12})=(0,-1/\phi,-\phi)$, $(x_{13},y_{13},z_{13})=(1/\phi,\phi,0)$, $(x_{14},y_{14},z_{14})=(-1/\phi,\phi,0)$, $(x_{15},y_{15},z_{15})=(1/\phi,-\phi,0)$, $(x_{16},y_{16},z_{16})=(-1/\phi,-\phi,0)$, $(x_{17},y_{17},z_{17})=(\phi,0,1/\phi)$, $(x_{18},y_{18},z_{18})=(-\phi,0,1/\phi)$, $(x_{19},y_{19},z_{19})=(\phi,0,-1/\phi)$, and $(x_{20},y_{20},z_{20})=(-\phi,0,-1/\phi)$, where $\phi=(1+\sqrt{5})/2$ is the golden ratio. Consider twenty additional equal masses $m_i=m$ for $i=21,\dots,40$ at the vertices of a second nested regular dodecahedron having positions $(x_{i+20},y_{i+20},z_{i+20})=\rho\,(x_i,y_i,z_i)$ for $i=1,\dots,20$ with $\rho>1$, and finally we consider masses $m_i=M$ for $i=41,\dots,60$ at the vertices of a third nested regular dodecahedron having positions $(x_{i+40},y_{i+40},z_{i+40})=R\,(x_i,y_i,z_i)$ for $i=1,\dots,20$ with $R>\rho$ (see Fig. 1(e)). Such configuration is central for the spatial 60–body problem when $m=m(R,\rho)$ and $M=M(R,\rho)$ are given by (4) with*



(a) The region $\mathcal{D}$.

(b) The level curves $m(R,\rho)=c$ and $M(R,\rho)=c$.

Fig. 8. The functions $m(R,\rho)$ and $M(R,\rho)$.

$$
\begin{aligned}
f(x,y)=&\frac{2(x-3y)}{(3x^2-2yx+3y^2)^{3/2}}-\frac{2(x+3y)}{(3x^2+2yx+3y^2)^{3/2}}+\\
&\frac{\left(5+\sqrt{5}\right)x-6\phi y}{2\phi\left(3x^2-2\sqrt{5}yx+3y^2\right)^{3/2}}-\frac{\left(5+\sqrt{5}\right)x+6\phi y}{2\phi\left(3x^2+2\sqrt{5}yx+3y^2\right)^{3/2}}+\\
&\frac{4xy}{3\sqrt{3}\left(x^2-y^2\right)^2}\ ,
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
g(x,y)=&\frac{2(3x-y)}{(3x^2-2yx+3y^2)^{3/2}}+\frac{2(3x+y)}{(3x^2+2yx+3y^2)^{3/2}}+\\
&\frac{6\phi x-\left(5+\sqrt{5}\right)y}{2\phi\left(3x^2-2\sqrt{5}yx+3y^2\right)^{3/2}}+\frac{6\phi x+\left(5+\sqrt{5}\right)y}{2\phi\left(3x^2+2\sqrt{5}yx+3y^2\right)^{3/2}}+\\
&\frac{2\left(x^2+y^2\right)}{3\sqrt{3}\left(x^2-y^2\right)^2}\ ,
\end{aligned}
$$

$\beta=\left(18+9\sqrt{2}+\sqrt{3}+9\sqrt{5}\right)/36$ *and* $(R,\rho)\in\mathcal{D}=\{(R,\rho)\in\mathbb{R}^2\ :\ m(R,\rho)>0,\ M(R,\rho)>0,\ R>\rho>1\}$ *(see Fig. 7(a) for the plot of* $\mathcal{D}$*).*

PROOF. It is easy to check that the positions (x_i,y_i,z_i) and the values of the masses m_i with $i=1,\ldots,60$ have been taken so that the center of mass of the resulting 60–body problem is located at the origin.

We substitute the positions and the values of the masses into (2). After some computations we obtain that $ex_9=ex_{10}=ex_{11}=ex_{12}=ex_{29}=ex_{30}=ex_{31}=ex_{32}=ex_{49}=ex_{50}=ex_{51}=ex_{52}=ey_{17}=ey_{18}=ey_{19}=ey_{20}=ey_{37}=ey_{38}=ey_{39}=ey_{40}=ey_{57}=ey_{58}=ey_{59}=ey_{60}=ez_{13}=ez_{14}=ez_{15}=ez_{16}=ez_{33}=ez_{34}=ez_{35}=ez_{36}=ez_{53}=ez_{54}=ez_{55}=ez_{56}=0$, $ex_1=$

$ex_3 = ex_4 = ex_7 = ey_1 = ey_2 = ey_4 = ey_6 = ez_1 = ez_2 = ez_3 = ez_5 = -ex_2$, $ex_5 = ex_6 = ex_8 = ey_3 = ey_5 = ey_7 = ey_8 = ez_4 = ez_6 = ez_7 = ez_8 = ex_2$, $ex_{13} = ex_{15} = ey_9 = ey_{11} = ez_{17} = ez_{18} = -ex_2/\phi$, $ex_{14} = ex_{16} = ey_{10} = ey_{12} = ez_{19} = ez_{20} = ex_2/\phi$, $ex_{17} = ex_{19} = ey_{13} = ey_{14} = ez_9 = ez_{10} = -ex_2\,\phi$, $ex_{18} = ex_{20} = ey_{15} = ey_{16} = ez_{11} = ez_{12} = ex_2\,\phi$, $ex_{21} = ex_{23} = ex_{24} = ex_{27} = ey_{21} = ey_{22} = ey_{24} = ey_{26} = ez_{21} = ez_{22} = ez_{23} = ez_{25} = -ex_{22}$, $ex_{25} = ex_{26} = ex_{28} = ey_{23} = ey_{25} = ey_{27} = ey_{28} = ez_{24} = ez_{26} = ez_{27} = ez_{28} = ex_{22}$, $ex_{33} = ex_{35} = ey_{29} = ey_{31} = ez_{37} = ez_{38} = -ex_{22}/\phi$, $ex_{34} = ex_{36} = ey_{30} = ey_{32} = ez_{39} = ez_{40} = ex_{22}/\phi$, $ex_{37} = ex_{39} = ey_{33} = ey_{34} = ez_{29} = ez_{30} = -ex_{22}\,\phi$, $ex_{38} = ex_{40} = ey_{35} = ey_{36} = ez_{31} = ez_{32} = ex_{22}\,\phi$, $ex_{41} = ex_{43} = ex_{44} = ex_{47} = ey_{41} = ey_{42} = ey_{44} = ey_{46} = ez_{41} = ez_{42} = ez_{43} = ez_{45} = -ex_{42}$, $ex_{45} = ex_{46} = ex_{48} = ey_{43} = ey_{45} = ey_{47} = ey_{48} = ez_{44} = ez_{46} = ez_{47} = ez_{48} = ex_{42}$, $ex_{53} = ex_{55} = ey_{49} = ey_{51} = ez_{57} = ez_{58} = -ex_{42}/\phi$, $ex_{54} = ex_{56} = ey_{50} = ey_{52} = ez_{59} = ez_{60} = ex_{42}/\phi$, $ex_{57} = ex_{59} = ey_{53} = ey_{54} = ez_{49} = ez_{50} = -ex_{42}\,\phi$, and $ex_{58} = ex_{60} = ey_{55} = ey_{56} = ez_{51} = ez_{52} = ex_{42}\,\phi$. System (2) is equivalent to system

$$
\begin{aligned}
ex_2 &= -\beta + \lambda + m\,f(\rho,1) + M f(R,1) = 0\ ,\\
ex_{22} &= -g(\rho,1) + \lambda\rho - \frac{m\beta}{\rho^2} + M f(R,\rho) = 0\ ,\\
ex_{42} &= -g(R,1) + \lambda R - m\,g(R,\rho) - \frac{M\beta}{R^2} = 0\ ,
\end{aligned}
\tag{14}
$$

with f and g given by (13) and $\beta = \left(18 + 9\sqrt{2} + \sqrt{3} + 9\sqrt{5}\right)/36$. So system (14) can be written as system (3). By using Proposition 5 we prove that system (14) has a unique solution $\lambda = \lambda(R,\rho)$, $m = m(R,\rho)$ and $M = M(R,\rho)$ defined for $R > \rho > 1$. This solution gives a central configuration of the 60–body problem if and only if R and ρ are such that $\lambda(R,\rho) > 0$, $m(R,\rho) > 0$ and $M(R,\rho) > 0$.

First we prove that f and g satisfy the conditions of Proposition 5. Since equations $3x^2 \pm 2yx + 3y^2 = 0$ and $3x^2 \pm 2\sqrt{5}yx + 3y^2 = 0$ have no real solutions and $x > y$, f and g are well defined for all $(x,y) \in D$. It is clear from the definition of g that $g(x,y) > 0$ for all $(x,y) \in D$. As we have seen in the proof of Proposition 6, in order to prove that $f(x,y) > 0$ in D it is sufficient to prove that $f(x,y)$ restricted to the half–line $x = a\,y$ with $y \geqslant 1$ is positive for all $a > 1$. It is easy to check that $f(a\,y, y) = f_1(a)/(9y^2)$ where

$$
\begin{aligned}
f_1(a) = {} & \frac{4\sqrt{3}a}{(a^2-1)^2} + \frac{18(a-3)}{(3a^2-2a+3)^{3/2}} - \frac{18(a+3)}{(3a^2+2a+3)^{3/2}} + \\
& \frac{9\left(\left(5+\sqrt{5}\right)a - 6\phi\right)}{2\phi\left(3a^2-2\sqrt{5}a+3\right)^{3/2}} - \frac{9\left(\left(5+\sqrt{5}\right)a + 6\phi\right)}{2\phi\left(3a^2+2\sqrt{5}a+3\right)^{3/2}}\ .
\end{aligned}
$$

We solve equation $f_1(a) = 0$ by using the procedure described in the Appendix and we see that it has no real solutions for $a > 1$. In particular $f_1(a) > 0$ for $a > 1$. Therefore condition (i) of Proposition 5 is satisfied.

Let $F(R,\rho) = f(R,\rho) - \rho f(R,1)$. As we have seen in the proof of Proposition 6, in order to prove that $F(R,\rho) > 0$ for all $R > \rho > 1$ it is sufficient to prove that $F(R,\rho)$ restricted to the half–line $\rho = a\,R$ with $R > 1/a$ is positive for all $0 < a < 1$. Since $0 < a < 1$ and $R > 1$, the inequality $F(R, a\,R) > 0$ is equivalent to the inequality $f_1(a) - f_2(R) > 0$ where

$$
\begin{aligned}
f_1(a) = & \frac{4\sqrt{3}}{(a^2-1)^2} - \frac{18(3a-1)}{a\,(3a^2-2a+3)^{3/2}} - \frac{18(3a+1)}{a\,(3a^2+2a+3)^{3/2}} - \\
& \frac{9\left(6\phi a - \sqrt{5} - 5\right)}{2\phi a\left(3a^2 - 2\sqrt{5}a + 3\right)^{3/2}} - \frac{9\left(6\phi a + \sqrt{5} + 5\right)}{2\phi a\left(3a^2 + 2\sqrt{5}a + 3\right)^{3/2}}\,, \\
f_2(R) = & 9R^3\left(\frac{2(R-3)}{(3R^2-2R+3)^{3/2}} - \frac{2(R+3)}{(3R^2+2R+3)^{3/2}} + \right. \\
& \frac{\left(5+\sqrt{5}\right)R - 6\phi}{2\phi\left(3R^2 - 2\sqrt{5}R + 3\right)^{3/2}} - \\
& \left. \frac{\left(5+\sqrt{5}\right)R + 6\phi}{2\phi\left(3R^2 + 2\sqrt{5}R + 3\right)^{3/2}} + \frac{4R}{3\sqrt{3}\,(R^2-1)^2}\right)\,.
\end{aligned}
$$

Now we see that f_2 is decreasing for $R > 1$. Indeed

$$
\begin{aligned}
f_2'(R) = & -\frac{16\sqrt{3}R^3}{(R^2-1)^3} + \frac{18\,(3R^3 - 5R^2 + 21R - 27)\,R^2}{(3R^2-2R+3)^{5/2}} - \\
& \frac{18\,(3R^3 + 5R^2 + 21R + 27)\,R^2}{(3R^2+2R+3)^{5/2}} + \\
& \frac{9\left(3\left(5+\sqrt{5}\right)R^3 - 50\phi R^2 + 21\left(5+\sqrt{5}\right)R - 54\phi\right)R^2}{2\phi\left(3R^2 - 2\sqrt{5}R + 3\right)^{5/2}} - \\
& \frac{9\left(3\left(5+\sqrt{5}\right)R^3 + 50\phi R^2 + 21\left(5+\sqrt{5}\right)R + 54\phi\right)R^2}{2\phi\left(3R^2 + 2\sqrt{5}R + 3\right)^{5/2}}\,.
\end{aligned}
$$

We solve equation $f_2'(R) = 0$ by using the procedure described in the Appendix and we see that it has no real solutions with $R > 1$. Moreover $f_2'(R) < 0$ for all $R > 1$, so f_2 is decreasing in this region. On the other hand $\lim_{R\to 1/a} f_2(R) = f(a)$. So $f_2(R) < f(a)$ for all $R > 1/a$ and consequently condition (ii) of Proposition 5 is satisfied. Condition (iii) can be proved directly from the definitions of f and g.

Finally we solve the equation $\beta x - g(x,1) = 0$ (see the Appendix) and we find a unique real solution with $x > 1$ which is $x = \alpha = 1.462226054217\ldots$. In particular $\beta x - g(x,1) < 0$ for $1 < x < \alpha$, and $\beta x - g(x,1) > 0$ for $x > \alpha$. So condition (iv) of Proposition 5 is satisfied.

Proposition 5 assures that there exist two nonempty sets $\mathcal{D}_1, \mathcal{D}_2 \subset \{(R,\rho) \in \mathbb{R}^2 : R > \rho > 1\}$ such that system (14) has a unique solution $\lambda = \lambda(R,\rho)$, $m = m(R,\rho)$ and $M = M(R,\rho)$ defined for all $R > \rho > 1$, satisfying that $m(R,\rho) > 0$ for all $(R,\rho) \in \mathcal{D}_1$, and $M(R,\rho) > 0$ for all $(R,\rho) \in \mathcal{D}_2$. Moreover $\mathcal{D}_1$ and $\mathcal{D}_2$ meet the boundary $R = \rho$ at the point $R = \rho = \alpha$. In Fig. 7(a) we plot the curves $m(R,\rho) = 0$ and $M(R,\rho) = 0$ and the region $\mathcal{D} = \mathcal{D}_1 \cap \mathcal{D}_2 = \{(R,\rho) \in \mathbb{R}^2 : m(R,\rho) > 0, M(R,\rho) > 0, R > \rho > 1\}$. It only remains to prove that $\lambda(R,\rho) > 0$ in $\mathcal{D}$. Since $\lambda = \lambda(R,\rho)$, $m = m(R,\rho)$ and $M = M(R,\rho)$ is solution of (14) and m and M are positive in $\mathcal{D}$, from equation $ex_{42} = 0$ of (14) we get that λ is positive in $\mathcal{D}$.

In Fig. 7(b) we plot the level curves $m(R,\rho) = c$ and $M(R,\rho) = c$ for $c = 0, 0.1, 1, 2, 3, 5, 8, 15, 30, 100, 200$. We note that given $c_1, c_2 > 0$ Fig. 7(b) shows that apparently there exists a unique intersection point between the level curves $m(R,\rho) = c_1$ and $M(R,\rho) = c_2$. Therefore it seems that for each pair of values of $m > 0$ and $M > 0$ there exists a unique $(R,\rho) \in \mathcal{D}$ such that the configuration is central. That is, for each pair of values of $m > 0$ and $M > 0$, we claim that Proposition 10 gives a unique central configuration of the spatial 60–body problem with the masses located at the vertices of three nested regular dodecahedra. This provides numerical evidence that Conjecture 2 holds for the regular dodecahedra.

8 Nested central configurations for the spatial pn–body problem with equal masses

We consider pn equal masses located at the vertices of p nested regular polyhedra of n vertices each one. Taking conveniently the unit of mass and length we can assume that all the masses are equal to one, and that the edges of the inner polyhedra have length 2. Let ρ_i denote the scale factor of the $i+1$–nested polyhedra for $i = 1, \ldots, p-1$. We have computed numerically the solution of system (2) for those kind of configurations when $p = 2, \ldots, 10$. The results that we have obtained are listed in the next table. In short we show that the Conjecture 3 maid at the introduction is true at least for $p = 4, \ldots, 10$.

	p	Scale factors
Tetrahedra	2	$\rho_1 = 2.24983$
	3	$\rho_1 = 2.14726$, $\rho_2 = 3.37391$
	4	$\rho_1 = 2.10294$, $\rho_2 = 3.1938$, $\rho_3 = 4.43301$
	5	$\rho_1 = 2.07799$, $\rho_2 = 3.10793$, $\rho_3 = 4.1896$, $\rho_4 = 5.44962$
	6	$\rho_1 = 2.06196$, $\rho_2 = 3.05684$, $\rho_3 = 4.06604$, $\rho_4 = 5.1521$, $\rho_5 = 6.43529$
	7	$\rho_1 = 2.05082$, $\rho_2 = 3.02283$, $\rho_3 = 3.9896$, $\rho_4 = 4.99421$, $\rho_5 = 6.09008$, $\rho_6 = 7.39691$
	8	$\rho_1 = 2.04264$, $\rho_2 = 2.99857$, $\rho_3 = 3.93732$, $\rho_4 = 4.89364$, $\rho_5 = 5.90068$, $\rho_6 = 7.00883$, $\rho_7 = 8.33905$
	9	$\rho_1 = 2.03639$, $\rho_2 = 2.9804$, $\rho_3 = 3.89926$, $\rho_4 = 4.8234$, $\rho_5 = 5.77724$, $\rho_6 = 6.79023$, $\rho_7 = 7.91184$, $\rho_8 = 9.26489$
	10	$\rho_1 = 2.03146$, $\rho_2 = 2.9663$, $\rho_3 = 3.87031$, $\rho_4 = 4.77141$, $\rho_5 = 5.68953$, $\rho_6 = 6.64511$, $\rho_7 = 7.666$, $\rho_8 = 8.80156$, $\rho_9 = 10.1768$
Octahedra	2	$\rho_1 = 1.99671$
	3	$\rho_1 = 1.90131$, $\rho_2 = 2.83266$
	4	$\rho_1 = 1.85771$, $\rho_2 = 2.67503$, $\rho_3 = 3.58738$
	5	$\rho_1 = 1.83201$, $\rho_2 = 2.59618$, $\rho_3 = 3.38335$, $\rho_4 = 4.28983$
	6	$\rho_1 = 1.81485$, $\rho_2 = 2.5474$, $\rho_3 = 3.27522$, $\rho_4 = 4.04863$, $\rho_5 = 4.95471$
	7	$\rho_1 = 1.8025$, $\rho_2 = 2.51384$, $\rho_3 = 3.20598$, $\rho_4 = 3.91551$, $\rho_5 = 4.68238$, $\rho_6 = 5.59079$
	8	$\rho_1 = 1.79315$, $\rho_2 = 2.48917$, $\rho_3 = 3.15721$, $\rho_4 = 3.82801$, $\rho_5 = 4.52743$, $\rho_6 = 5.29155$, $\rho_7 = 6.20377$
	9	$\rho_1 = 1.7858$, $\rho_2 = 2.4702$, $\rho_3 = 3.12075$, $\rho_4 = 3.76522$, $\rho_5 = 4.42347$, $\rho_6 = 5.11718$, $\rho_7 = 5.88078$, $\rho_8 = 6.79766$
	10	$\rho_1 = 1.77987$, $\rho_2 = 2.45512$, $\rho_3 = 3.09234$, $\rho_4 = 3.71763$, $\rho_5 = 4.34775$, $\rho_6 = 4.99826$, $\rho_7 = 5.6889$, $\rho_8 = 6.45334$, $\rho_9 = 7.37538$
Cube	2	$\rho_1 = 1.87207$
	3	$\rho_1 = 1.78431$, $\rho_2 = 2.5734$
	4	$\rho_1 = 1.74307$, $\rho_2 = 2.43205$, $\rho_3 = 3.19068$
	5	$\rho_1 = 1.7182$, $\rho_2 = 2.35971$, $\rho_3 = 3.01117$, $\rho_4 = 3.75451$
	6	$\rho_1 = 1.70125$, $\rho_2 = 2.3141$, $\rho_3 = 2.91407$, $\rho_4 = 3.54555$, $\rho_5 = 4.28027$
	7	$\rho_1 = 1.68883$, $\rho_2 = 2.28217$, $\rho_3 = 2.85082$, $\rho_4 = 3.428$, $\rho_5 = 4.04738$, $\rho_6 = 4.77706$
	8	$\rho_1 = 1.67927$, $\rho_2 = 2.25834$, $\rho_3 = 2.8056$, $\rho_4 = 3.34953$, $\rho_5 = 3.91248$, $\rho_6 = 4.52403$, $\rho_7 = 5.25077$
	9	$\rho_1 = 1.67164$, $\rho_2 = 2.23975$, $\rho_3 = 2.77132$, $\rho_4 = 3.29244$, $\rho_5 = 3.82065$, $\rho_6 = 4.37405$, $\rho_7 = 4.98035$, $\rho_8 = 5.70551$
	10	$\rho_1 = 1.6654$, $\rho_2 = 2.22478$, $\rho_3 = 2.74428$, $\rho_4 = 3.24862$, $\rho_5 = 3.75289$, $\rho_6 = 4.27034$, $\rho_7 = 4.81706$, $\rho_8 = 5.41978$, $\rho_9 = 6.14427$

Icosahedra	2	$\rho_1 = 1.7373$
	3	$\rho_1 = 1.65864,\ \rho_2 = 2.29946$
	4	$\rho_1 = 1.62055,\ \rho_2 = 2.17682,\ \rho_3 = 2.77846$
	5	$\rho_1 = 1.597,\ \rho_2 = 2.11241,\ \rho_3 = 2.62631,\ \rho_4 = 3.20555$
	6	$\rho_1 = 1.58059,\ \rho_2 = 2.07091,\ \rho_3 = 2.54204,\ \rho_4 = 3.03173,\ \rho_5 = 3.59618$
	7	$\rho_1 = 1.56832,\ \rho_2 = 2.04131,\ \rho_3 = 2.48607,\ \rho_4 = 2.93173,\ \rho_5 = 3.40547,\ \rho_6 = 3.95936$
	8	$\rho_1 = 1.55871,\ \rho_2 = 2.01883,\ \rho_3 = 2.44535,\ \rho_4 = 2.86376,\ \rho_5 = 3.29262,\ \rho_6 = 3.75496,\ \rho_7 = 4.30089$
	9	$\rho_1 = 1.55092,\ \rho_2 = 2.00101,\ \rho_3 = 2.414,\ \rho_4 = 2.81352,\ \rho_5 = 3.21447,\ \rho_6 = 3.63129,\ \rho_7 = 4.08504,\ \rho_8 = 4.62477$
	10	$\rho_1 = 1.54443,\ \rho_2 = 1.98646,\ \rho_3 = 2.38892,\ \rho_4 = 2.77439,\ \rho_5 = 3.15594,\ \rho_6 = 3.54436,\ \rho_7 = 3.9521,\ \rho_8 = 4.39915,\ \rho_9 = 4.93391$
Dodecahedra	2	$\rho_1 = 1.62944$
	3	$\rho_1 = 1.56416,\ \rho_2 = 2.08214$
	4	$\rho_1 = 1.53149,\ \rho_2 = 1.98208,\ \rho_3 = 2.45611$
	5	$\rho_1 = 1.51074,\ \rho_2 = 1.92829,\ \rho_3 = 2.33363,\ \rho_4 = 2.78188$
	6	$\rho_1 = 1.49595,\ \rho_2 = 1.89291,\ \rho_3 = 2.26444,\ \rho_4 = 2.64358,\ \rho_5 = 3.07425$
	7	$\rho_1 = 1.48467,\ \rho_2 = 1.8672,\ \rho_3 = 2.21768,\ \rho_4 = 2.56261,\ \rho_5 = 2.92411,\ \rho_6 = 3.34176$
	8	$\rho_1 = 1.47568,\ \rho_2 = 1.84735,\ \rho_3 = 2.18313,\ \rho_4 = 2.5067,\ \rho_5 = 2.83378,\ \rho_6 = 3.18238,\ \rho_7 = 3.58985$
	9	$\rho_1 = 1.46826,\ \rho_2 = 1.83139,\ \rho_3 = 2.15614,\ \rho_4 = 2.46478,\ \rho_5 = 2.77033,\ \rho_6 = 3.0844,\ \rho_7 = 3.42301,\ \rho_8 = 3.82225$
	10	$\rho_1 = 1.46201,\ \rho_2 = 1.81816,\ \rho_3 = 2.13425,\ \rho_4 = 2.43172,\ \rho_5 = 2.72218,\ \rho_6 = 3.0146,\ \rho_7 = 3.31866,\ \rho_8 = 3.64924,\ \rho_9 = 4.04162$

Notice from the table that the size of the polyhedra at the same nested level decreases when increasing p and when increasing the number of vertices of the regular polyhedra.

9 Proof of Theorem 4

We consider $3n$ masses at the vertices of three nested polyhedra with the same number of vertices as in Propositions 6, 7, 8, 9 and 10. We consider and additional mass $m_0 = \mu$ located at the origin. It is easy to cheek that the equations of the spatial central configurations (1) for the $3n+1$–body problem with the mass m_0 located at the origin are

$$
\begin{aligned}
nex_0 &= \sum_{j=1}^{N} -\frac{m_j\, x_j}{(x_j^2+y_j^2+z_j^2)^{3/2}} = 0\ , \\
ney_0 &= \sum_{j=1}^{N} -\frac{m_j\, y_j}{(x_j^2+y_j^2+z_j^2)^{3/2}} = 0\ , \\
nez_0 &= \sum_{j=1}^{N} -\frac{m_j\, z_j}{(x_j^2+y_j^2+z_j^2)^{3/2}} = 0\ , \\
nex_i &= ex_i + \frac{\mu\, x_i}{(x_j^2+y_j^2+z_j^2)^{3/2}} = 0\ , \\
ney_i &= ey_i + \frac{\mu\, y_i}{(x_j^2+y_j^2+z_j^2)^{3/2}} = 0\ , \\
nez_i &= ez_i + \frac{\mu\, z_i}{(x_j^2+y_j^2+z_j^2)^{3/2}} = 0\ ,
\end{aligned} \tag{15}
$$

for $i = 1, \ldots, N$, where ex_i, ey_i and ez_i are defined as in (2).

Notice that in Propositions 6, 7, 8, 9 and 10 the positions (x_i, y_i, z_i) and the values of the masses m_i have been taken so that the center of mass of the problem is located at the origin. Moreover all the vertices in the same polyhedron are at the same distance d from the origin. Therefore equations $nex_0 = 0$, $ney_0 = 0$ and $nez_0 = 0$ are always satisfied.

Proceeding in a similar way than in Sections 3, 4, 5, 6, and 7, we prove that, for the nested polyhedra configurations of the $3n+1$–body problem, system (15) can be reduced to a system of the form $A\,X = b$ where A and X are given as in (3),

$$
b = \left(\beta + \gamma\,\mu, g(\rho,1) + \frac{\gamma\,\mu}{\rho^2}, g(R,1) + \frac{\gamma\,\mu}{R^2}\right)\ ,
$$

and $\gamma = x_i/d^3 > 0$ for some i. Proceeding as in Section 2 we prove that Proposition 5 remains valid when we replace condition (iv) by the following condition: $\beta x + \gamma\,\mu(x^3-1)/x^2 - g(x,1) > 0$ for all $x > \alpha(\mu)$ and some $\alpha(\mu) > 1$, and $\beta x + \gamma\,\mu(x^3-1)/x^2 - g(x,1) < 0$ for $1 < x < \alpha(\mu)$.

Notice that the function $\beta x + \gamma\,\mu(x^3-1)/x^2$ is increasing and positive for all $x > 1$. On the other hand it is easy to check that the functions $g(x,1)$ are decreasing for all type of polyhedra and that $\lim_{x\to 1} g(x,1)/(1/(x-1)^2) = c$ for some constant c which depends on the chosen polyhedra. Moreover $\lim_{x\to 1} g(x,1) = +\infty$. Therefore equation $\beta x + \gamma\,\mu(x^3-1)/x^2 - g(x,1) = 0$ has a unique real solution $x = \alpha(\mu)$ with $\alpha(\mu) > 1$ for all $\mu \geqslant 0$ such that $\beta x + \gamma\,\mu(x^3-1)/x^2 - g(x,1) > 0$ for $x > \alpha(\mu)$, and $\beta x + \gamma\,\mu(x^3-1)/x^2 - g(x,1) < 0$ for $1 < x < \alpha(\mu)$. Moreover $\alpha(\mu) \to \alpha$ when $\mu \to 0$ and $\alpha(\mu) \to 1$ when $\mu \to +\infty$. In short we have proved Theorem 4.

Appendix

In this appendix we analyze the resolution of equations of the form $F(x) = 0$ when F is a rational function containing radicals. These type of equations are solved by following the next steps.

(1) We eliminate the fractions by multiplying equation $F(x) = 0$ by the least common denominator of $F(x)$.
(2) We eliminate the radicals of the resulting equation by isolating in a convenient way one or more radicals on one side of the equation and squaring both sides of the equation. If the resulting equation still contains radicals, then we repeat the process again. At the end we obtain a polynomial equation.
(3) We find numerically all the solutions of the polynomial equation obtained in steep (2).
(4) We cheek which of these solutions are really solutions of the initial equation $F(x) = 0$.

Now we detail how to group the radicals in steep (2) for each type of equations that appear in this work after applying steep (1).

(a) Equations with one radical: $\alpha_1\sqrt{a} + \alpha_2 = 0$. We eliminate the radicals by applying steep (2) in the following way

$$(\alpha_1\sqrt{a})^2 = (-\alpha_2)^2 \ .$$

(b) Equations of the form: $\alpha_1\sqrt{a} + \alpha_2\sqrt{b} + \alpha_3\sqrt{a}\sqrt{b} = 0$. Applying steep (2) in the following way

$$(\alpha_1\sqrt{a} + \alpha_2\sqrt{b})^2 = (-\alpha_3\sqrt{a}\sqrt{b})^2 \ ,$$

we obtain an equation with one radical of the form $\beta_1\sqrt{a}\sqrt{b} + \beta_2 = 0$.

(c) Equations of the form $\alpha_1\sqrt{b}\sqrt{c}\sqrt{d}+\alpha_2\sqrt{a}\sqrt{c}\sqrt{d}+\alpha_3\sqrt{a}\sqrt{b}\sqrt{d}+\alpha_4\sqrt{a}\sqrt{b}\sqrt{c}+\alpha_5\sqrt{a}\sqrt{b}\sqrt{c}\sqrt{d} = 0$. Applying steep (2) by grouping the terms in the following way

$$(\alpha_1\sqrt{b}\sqrt{c}\sqrt{d}+\alpha_2\sqrt{a}\sqrt{c}\sqrt{d})^2=(-\alpha_3\sqrt{a}\sqrt{b}\sqrt{d}-\alpha_4\sqrt{a}\sqrt{b}\sqrt{c}-\alpha_5\sqrt{a}\sqrt{b}\sqrt{c}\sqrt{d})^2,$$

we obtain an equation of the form $\beta_1\sqrt{c}+\beta_2\sqrt{d}+\beta_3\sqrt{a}\sqrt{b}+\beta_4\sqrt{c}\sqrt{d}+\beta_5 = 0$. Applying steep (2) to this equation in the following way

$$(\beta_1\sqrt{c} + \beta_2\sqrt{d} + \beta_4\sqrt{c}\sqrt{d} + \beta_5)^2 = (-\beta_3\sqrt{a}\sqrt{b})^2 \ ,$$

we obtain an equation with three radicals of the form $\gamma_1\sqrt{c} + \gamma_2\sqrt{d} + \gamma_3\sqrt{c}\sqrt{d} + \gamma_4 = 0$. Finally, applying steep (2) again in the following way

$$(\gamma_1\sqrt{c} + \gamma_2\sqrt{d})^2 = (-\gamma_3\sqrt{c}\sqrt{d} - \gamma_4)^2 \ ,$$

we obtain an equation with one radical.

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