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FAMILIES OF PERIODIC ORBITS FOR THE SPATIAL ISOSCELES 3-BODY PROBLEM*

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Abstract. We study the families of periodic orbits of the spatial isosceles 3-body problem (for small enough values of the mass lying on the symmetry axis) coming via the analytic continuation method from periodic orbits of the circular Sitnikov problem. Using the first integral of the angular momentum, we reduce the dimension of the phase space of the problem by two units. Since periodic orbits of the reduced isosceles problem generate invariant two-dimensional tori of the nonreduced problem, the analytic continuation of periodic orbits of the (reduced) circular Sitnikov problem at this level becomes the continuation of invariant two-dimensional tori from the circular Sitnikov problem to the nonreduced isosceles problem, each one filled with periodic or quasi-periodic orbits. These tori are not KAM tori but just isotropic, since we are dealing with a three-degrees-of-freedom system. The continuation of periodic orbits is done in two different ways, the first going directly from the reduced circular Sitnikov problem to the reduced isosceles problem, and the second one using two steps: first we continue the periodic orbits from the reduced circular Sitnikov problem to the reduced elliptic Sitnikov problem, and then we continue those periodic orbits of the reduced elliptic Sitnikov problem to the reduced isosceles problem. The continuation in one or two steps produces different results. This work is merely analytic and uses the variational equations in order to apply Poincaré's continuation method.

Key words. periodic orbits, quasi-periodic orbits, 3-body problem, analytic continuation method

AMS subject classifications. 70F15, 37N05

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1. Introduction. We consider a special case of the spatial 3-body problem, the *spatial isosceles* 3-*body problem*, or simply the *isosceles problem*. This problem consists of describing the motion of two equally massive bodies, $m_1 = m_2 = 1/2$, having initial conditions and velocities symmetric with respect to a straight line which passes through their center of mass, and a third body, with mass $m_3 = \mu$, having initial position and velocity on this straight line. This problem is called the isosceles problem because the three bodies form an isosceles triangle at any time, eventually degenerated to a segment.

The most interesting application of the spatial isosceles 3-body problem was given by Xia in [25]. He used two spatial isosceles 3-body problems to prove that five bodies can escape to infinity in a finite time without collision. Other works on the spatial isosceles 3-body problem are [16] and the references therein. If in the spatial isosceles 3-body problem the initial positions and velocities of the three bodies are contained in a plane, then the motion remains always in this plane, and we have the so-called *planar isosceles* 3-*body problem*. There are several papers about the planar isosceles 3-body problem, for instance, [9], [17], etc.

When the third body of the isosceles 3-body problem has infinitesimal mass (i.e., $\mu = 0$) then we obtain the *restricted isosceles problems*. Depending on the motion

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of the primaries m_1 and m_2 we have seven different cases for the restricted isosceles problems. Here we consider, due to their richness in periodic orbits, only the cases in which the primaries move in circular or elliptic orbits of the 2-body problem, the *circular and elliptic* restricted isosceles problems, also called the *circular and elliptic Sitnikov problems*.

The isosceles problem and the restricted isosceles problems possess the first integral of the angular momentum. In section 3 we will prove that the phase portrait of any of these problems on each level of the angular momentum c with $c \neq 0$ is the same. Notice that the angular momentum $c = 0$ contains the triple and the double collision orbits, but collision orbits are not treated in this work. With a fixed value of the angular momentum $c \neq 0$, we reduce by two dimensions (an angle and its derivative) the phase space of the isosceles problem, obtaining the *reduced isosceles problem*. In particular, we see that each periodic orbit of the reduced isosceles problem gives an invariant two-dimensional torus of the isosceles problem, filled with either periodic or quasi-periodic orbits, which is not a KAM tori. We note that the circular and elliptic Sitnikov problems that appear in the literature are essentially our reduced circular and elliptic Sitnikov problems.

The main objective of this work is to prove that the invariant two-dimensional tori of the restricted isosceles problem that come from the known periodic orbits of the reduced circular Sitnikov problem persist when we pass from the restricted isosceles problem to the isosceles problem for $\mu > 0$ sufficiently small. Consequently these tori persist inside the general spatial 3-body problem. The main tool for proving this result will be the classical Poincaré analytic continuation method of periodic orbits. In particular, we continue the known periodic orbits of the reduced circular Sitnikov problem to periodic orbits of the reduced isosceles problem for $\mu > 0$ sufficiently small. In order to do that, we will use the symmetries of the problem. The isosceles problem is invariant under the time reversibility (t-symmetry), and it is also invariant under a symmetry with respect to the plane defined by the motion of m_1 and m_2 (r-symmetry). These symmetries will allow us to find r- and t-symmetric periodic orbits for the reduced isosceles problem. We still distinguish another type of symmetric periodic orbits, the *doubly symmetric periodic orbits*, which are simultaneously r- and t-symmetric periodic orbits.

Using the analytical continuation method of Poincaré, we will continue the known periodic orbits of the reduced circular Sitnikov problem (where $\mu = 0$), which are doubly symmetric periodic orbits, to symmetric periodic orbits of the reduced isosceles problem for $\mu > 0$ sufficiently small. Those periodic orbits are continued in two different ways. The first goes directly from the reduced circular Sitnikov problem to the reduced isosceles problem. The second uses two steps: first we continue the periodic orbits from the reduced circular Sitnikov problem to symmetric periodic orbits of the reduced elliptic Sitnikov problem (where $\mu = 0$) for small values of the eccentricity e, and then we continue those symmetric periodic orbits of the reduced elliptic Sitnikov problem to the reduced isosceles problem for small values of $\mu > 0$.

A key point in this work is the knowledge of an analytical expression for the solution of the variational equations of the reduced circular (elliptic) Sitnikov problem along the periodic solution that we want to continue. We must remark that all results presented in this paper are analytical results.

The main results about continuation of periodic orbits from the reduced circular Sitnikov problem to the reduced isosceles problem are summarized in the following result.

THEOREM A. *Let γ be a periodic orbit of the reduced circular Sitnikov problem with period $T > \pi/\sqrt{2}$, and let $f(e) = (1-e^2)^{3/2}$. Then γ can be continued to the following families of periodic orbits of the reduced isosceles problem with angular momentum $c = 1/4$ and $\mu > 0$ sufficiently small:*

1. *Case $T = 2\pi\omega$ with $\omega > 1/(2\sqrt{2})$ an irrational number.*
   (a) *γ can be continued directly to one 2-parameter family (on μ and τ) of doubly symmetric periodic orbits with period τ sufficiently close to T.*
2. *Case $T = 2\pi p/q$ for some $p, q \in \mathbb{N}$ coprime with $p > q/(2\sqrt{2})$.*
   (a) *p odd:*
      i. *γ can be continued directly to one 2-parameter family (on μ and τ) of doubly symmetric periodic orbits with period τ sufficiently close to T.*
      ii. *γ can be continued by two steps to two 2-parameter families (on μ and e) of r-symmetric periodic orbits with period $qT\,f(e)$ where $e > 0$ is sufficiently small.*
      iii. *γ can be continued by two steps to two 2-parameter families (on μ and e) of t-symmetric periodic orbits with period $qT\,f(e)$ where $e > 0$ is sufficiently small.*
   (b) *p even and $q \neq 1$:*
      i. *γ can be continued directly to one 2-parameter family (on μ and τ) of doubly symmetric periodic orbits with period τ sufficiently close to T.*
      ii. *γ can be continued by two steps to two 2-parameter families (on μ and e) of doubly symmetric periodic orbits of period $qT\,f(e)$ where $e > 0$ is sufficiently small.*
   (c) *p even and $q = 1$:*
      i. *γ can be continued by two steps to two 2-parameter families (on μ and e) of doubly symmetric periodic orbits of period $qT\,f(e)$ where $e > 0$ is sufficiently small.*

Using direct continuation we can continue all periodic orbits of the reduced circular Sitnikov problem except the ones that have period multiple of 4π. In particular, we can continue the periodic orbits, with period $2\pi\omega$ and ω irrational. These periodic orbits become quasi-periodic orbits in the restricted isosceles problem. So, in fact we have continued quasi-periodic orbits of the restricted isosceles problem to either periodic or quasi-periodic orbits of the isosceles problem for $\mu > 0$ sufficiently small.

The continuation in two steps allows us to continue only periodic orbits of the reduced circular Sitnikov problem with period $T = 2\pi p/q$ for all $p, q \in \mathbb{N}$ coprime and $p > q/(2\sqrt{2})$. These periodic orbits become periodic orbits of the restricted isosceles problem. We note that the periodic orbits of the reduced circular Sitnikov problem that cannot be continued directly can be continued in two steps. Moreover the rest of the periodic orbits with period $T = 2\pi p/q$ can be continued in both ways, obtaining different periodic orbits for the reduced isosceles problem.

Since each periodic orbit of the reduced isosceles problem gives an invariant two-dimensional torus of the isosceles problem, in particular we have continued the invariant two-dimensional tori of the circular restricted isosceles problem ($\mu = 0$) to invariant two-dimensional tori of the isosceles problem for $\mu > 0$ sufficiently small. In section 13 we state Theorem A translated to the language of tori for the isosceles problem.

This paper is organized as follows. In section 2 we give the equations of motion of the isosceles problem in appropriate cylindrical coordinates; these coordinates will allow us to define the reduced isosceles problem in section 3. In section 4 we give the relationships between the orbits of the reduced isosceles problem and the isosceles problem. In particular, we see that if $\overline{\varphi}$ is an orbit for the reduced isosceles problem, then $\overline{\varphi} \times \mathbb{S}^1$ is an invariant manifold for the isosceles problem (for more details see Theorem 4.1). In section 5 we analyze the symmetries of the reduced isosceles problem. In section 6 we define the restricted isosceles problems and the reduced restricted isosceles problems. In this work, we will consider only the circular and elliptic restricted isosceles problems, which are treated in sections 7 and 8, respectively. In particular, we are interested in the invariant two-dimensional tori of these problems that come from periodic orbits of the corresponding reduced problems. In section 7.1, we summarize the basic properties given in [8] of the periodic solutions of the circular Sitnikov problem. In section 8.1 we summarize the basic properties of the periodic solutions of the elliptic Sitnikov problem and give the basic results on continuation of periodic solutions from the circular Sitnikov problem ($e = 0$) to the elliptic Sitnikov problem for $e > 0$ sufficiently small. These results have also been extracted from [8]. In section 9 we analyze the variational equations of the reduced circular and elliptic Sitnikov problem and explicitly give the solution of the variational equations of the Kepler problem along a circular or elliptic periodic solution and the solution of the variational equations of the circular Sitnikov problem. In section 10 we analyze the direct continuation of periodic solutions from the reduced circular Sitnikov problem to the isosceles problem for $\mu > 0$ sufficiently small; in particular, we prove statements 1(a), 2(a)i, and 2(b)i of Theorem A (see Theorem 10.1). In section 11 we analyze the continuation of the symmetric periodic solutions of the reduced elliptic Sitnikov problem that we give in section 8 to the isosceles problem for $\mu > 0$ sufficiently small. The continuation by two steps from the reduced circular Sitnikov problem to the reduced isosceles problem is analyzed in section 12; in particular, we prove the remaining statements of Theorem A (see Theorem 12.8). In section 13 we summarize the basic results on continuation of invariant two-dimensional tori from the circular restricted isosceles problem to the isosceles problem for $\mu > 0$ small.

2. Coordinates and equations of motion of the isosceles problem. Let P_1 and P_2 be two particles, with equal masses $m_1 = m_2$, having initial positions and velocities symmetric with respect to a straight line that passes through their center of mass. Let P_3 be a third particle, with mass m_3, having initial position and velocity on this straight line. The *spatial isosceles* 3*-body problem*, or simply the *isosceles problem* in this work, consists of describing the motion of these three particles under their mutual Newtonian gravitational attraction. We note that the solutions of the isosceles problem are in fact solutions of the general spatial 3-body problem.

We choose an inertial coordinate system (X, Y, Z) in such a way that the Z-axis is the straight line that contains the particle P_3. The initial positions of the particles P_1, P_2, and P_3 in this coordinate system are (X, Y, Z_2), $(-X, -Y, Z_2)$, $(0, 0, Z_1)$, respectively, and their respective velocities are $(\dot{X}, \dot{Y}, \dot{Z}_2)$, $(-\dot{X}, -\dot{Y}, \dot{Z}_2)$, and $(0, 0, \dot{Z}_1)$ (see Figure 2.1). Of course, the dot denotes the derivative with respect to the time t.

In order to develop our analysis we will use the cylindrical coordinates $(r, z, \theta) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{S}^1$ introduced as follows. Here $\mathbb{R}^+$ denotes the open interval $(0, \infty)$. First we put the origin $\mathbf{0}$ of the coordinate system at the center of mass of m_1, m_2, and m_3, which implies taking $Z_2 = -m_3 Z_1$. Then we define a new variable $z = Z_1 - Z_2 \in \mathbb{R}$ which denotes the distance between the third particle P_3 and the orthogonal plane to the Z-axis that contains the particles P_1 and P_2 with the convenient sign (positive

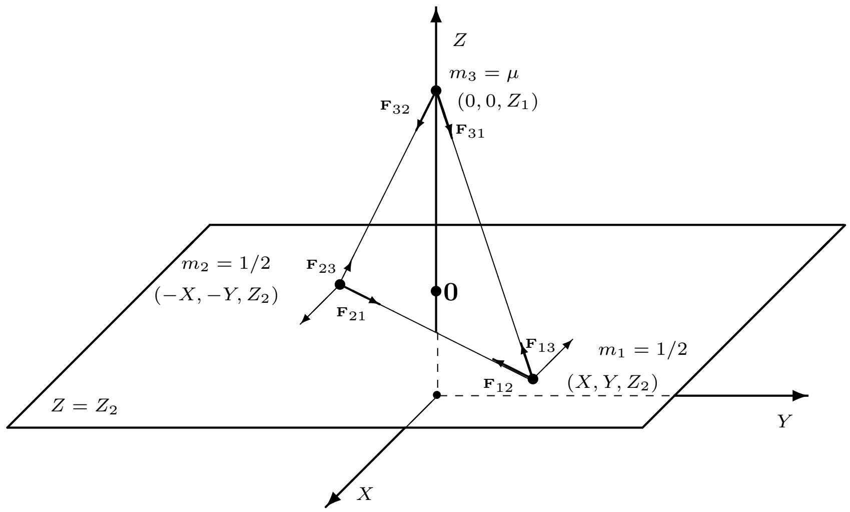

FIG. 2.1. *The isosceles problem.*

if $Z_1 > Z_2$ and negative if $Z_1 < Z_2$). Finally we consider polar coordinates, $(r, \theta) \in \mathbb{R}^+ \times \mathbb{S}^1$, in the above orthogonal plane by taking $X = r\cos\theta$ and $Y = r\sin\theta$.

We choose the unit of mass in such a way that $m_1 = m_2 = 1/2$ and $m_3 = \mu$, and the unit of length is chosen so that the gravitational constant is one. Then the *kinetic energy* and the *potential energy* in the coordinate system $(r, \dot r, z, \dot z, \theta, \dot\theta)$ are given, respectively, by

$$T = \frac{1}{2}\left(\dot r^2 + r^2\dot\theta^2 + \frac{\mu}{1+\mu}\,\dot z^2\right) \qquad \text{and} \qquad U = -\frac{1}{8\,r} - \frac{\mu}{(z^2+r^2)^{1/2}}.$$

Therefore the *Lagrangian equations* of motion for the isosceles problem are

$$\begin{aligned} \frac{d}{dt}(\dot r) &= r\dot\theta^2 - \frac{1}{8\,r^2} - \frac{\mu r}{(z^2+r^2)^{3/2}}\ , \\ \frac{d}{dt}\left(\frac{\mu}{1+\mu}\,\dot z\right) &= -\frac{\mu z}{(z^2+r^2)^{3/2}}\ , \\ \frac{d}{dt}(r^2\dot\theta) &= 0. \end{aligned} \tag{2.1}$$

We note that the third equation of system (2.1) can be integrated directly, obtaining the first integral of the *angular momentum*

$$C = r^2\dot\theta. \tag{2.2}$$

Of course, system (2.1) also has the first integral given by the *energy* $H = T + U$.

3. The reduced isosceles problem. To avoid singular situations, throughout this work we consider only solutions of system (2.1) having nonzero angular momentum (i.e., in particular, we do not consider solutions with collision between the masses, either triple or double). We note that under this assumption it is sufficient to consider solutions of (2.1) having a fixed value of the angular momentum $C = c$ for some $c \neq 0$, because the phase portrait of the isosceles problem on each angular momentum level $c \neq 0$ is the same as that shown in the following proposition.

PROPOSITION 3.1. *Let* $(r(t), \dot r(t), z(t), \dot z(t), \theta(t), \dot\theta(t))$ *be a solution of the isosceles problem* (2.1) *with angular momentum* $C = c$ *for some* $c \neq 0$. *If we take* $\alpha^{1/2} = \overline{c}/c \neq 0$, *then*

$$\varphi(t) = \left(\alpha r(\alpha^{3/2}t), \frac{\dot r(\alpha^{3/2}t)}{\alpha^{1/2}}, \alpha z(\alpha^{3/2}t), \frac{\dot z(\alpha^{3/2}t)}{\alpha^{1/2}}, \theta(\alpha^{3/2}t), \frac{\dot\theta(\alpha^{3/2}t)}{\alpha^{3/2}}\right)$$

is a solution of (2.1) *with angular momentum* $\overline{c}$.

Proof. It is easy to see that system (2.1) is invariant under the transformation

$$(t, r, \dot r, z, \dot z, \theta, \dot\theta) \longmapsto (\alpha^{3/2}t, \alpha r, \alpha^{-1/2}\dot r, \alpha z, \alpha^{-1/2}\dot z, \theta, \alpha^{-3/2}\dot\theta).$$

Thus $\varphi(t)$ is a solution of (2.1). Moreover the angular momentum of $\varphi(t)$ is given by

$$\alpha^2 r^2(\alpha^{3/2}t)\alpha^{-3/2}\dot\theta(\alpha^{3/2}t) = \alpha^{1/2}c = \overline{c}.$$

Then $\varphi(t)$ is a solution of (2.1) with angular momentum $\overline{c}$. ∎

Assuming that the value of the angular momentum is fixed at $C = c$ for some $c \neq 0$, we can reduce by two units the dimension of the phase space. Indeed, the variable θ does not appear explicitly in system (2.1); moreover from (2.2), $\dot\theta = c/r^2$, and thus we need to consider only the first two equations of (2.1) with $\dot\theta$ replaced by c/r^2. That is, we need to consider only the *reduced isosceles problem*

$$\ddot r = \frac{c^2}{r^3} - \frac{1}{8\,r^2} - \frac{\mu r}{(z^2 + r^2)^{3/2}}\ , \qquad \ddot z = -\frac{(1+\mu)z}{(z^2+r^2)^{3/2}}. \tag{3.1}$$

4. Relationships between the reduced isosceles problem and the isosceles problem. Let $\varphi(t) = (r(t), \dot r(t), z(t), \dot z(t))$ be a solution of the reduced isosceles problem (3.1) for a fixed $c \neq 0$ with initial conditions $r(0) = r_0$, $\dot r(0) = \dot r_0$, $z(0) = z_0$, $\dot z(0) = \dot z_0$. For each $\theta_0 \in \mathbb{S}^1$, the solution $\varphi(t)$ gives rise to a solution $\gamma_{\varphi,\theta_0,c}(t) = (r(t), \dot r(t), z(t), \dot z(t), \theta(t), \dot\theta(t))$ of the isosceles problem (2.1) with angular momentum c, having initial conditions $r(0) = r_0$, $\dot r(0) = \dot r_0$, $z(0) = z_0$, $\dot z(0) = \dot z_0$, $\theta(0) = \theta_0 \pmod{2\pi}$, $\dot\theta(0) = c/r_0^2$, where $\dot\theta(t)$ and $\theta(t)$ are given by

$$\dot\theta(t) = \frac{c}{r^2(t)} \qquad \text{and} \qquad \theta(t) = \int_0^t \frac{c}{r^2(\tau)}\, d\tau + \theta_0 = F(t) + \theta_0. \tag{4.1}$$

It is well known that all solutions of the 3-body problem, except those that end in collision, are defined for all $t \in \mathbb{R}$ (see [18] or [21]). Since our isosceles problem is a particular case of the general 3-body problem, all its solutions with angular momentum $c \neq 0$ are defined for all $t \in \mathbb{R}$.

Fixing a value of $c \neq 0$, the union of the orbits $\gamma_{\varphi,\theta_0,c} = \{\gamma_{\varphi,\theta_0,c}(t) \,:\, t \in \mathbb{R}\}$, varying $\theta_0 \in \mathbb{S}^1$, is an invariant submanifold $\mathcal{E}_{\varphi,c}$ of the phase space of the isosceles problem $\mathcal{E} = \{(r, \dot r, z, \dot z, \theta, \dot\theta) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{S}^1 \times \mathbb{R}\}$. In particular, $\mathcal{E}_{\varphi,c}$ is an invariant submanifold of $\mathcal{E}_c = \{(r, \dot r, z, \dot z, \theta, \dot\theta) \in \mathcal{E} \,:\, r^2\dot\theta = c\}$. Note that $\mathcal{E}_c$, called the *angular momentum level* $C = c$, is a submanifold of dimension 5 of $\mathcal{E}$ because $c \neq 0$. The invariant submanifold $\mathcal{E}_{\varphi,c}$ is called the *relative set* associated to the orbit $\overline{\varphi} = \{\varphi(t) :\ t \in \mathbb{R}\}$, and it is diffeomorphic to $\overline{\varphi} \times \mathbb{S}^1$.

By the qualitative theory of differential equations we know that the orbits of the reduced isosceles problem (3.1) can be either equilibrium points, periodic orbits, or orbits diffeomorphic to $\mathbb{R}$. Thus if $\overline{\varphi}$ is an equilibrium point, then the corresponding relative set is diffeomorphic to a circle $\mathbb{S}^1$ (a *relative periodic orbit*). If $\overline{\varphi}$ is a periodic orbit (i.e., a closed curve diffeomorphic to $\mathbb{S}^1$), then the corresponding relative set is diffeomorphic to a two-dimensional torus $\mathbb{S}^1 \times \mathbb{S}^1$ (a *relative torus*). This relative torus can be filled with either periodic or quasi-periodic orbits (in this last case the orbits are dense on the torus). We note that these kinds of tori are not KAM tori (see, for instance, [1]), because they are two-dimensional tori of a problem with three degrees of freedom, and the KAM tori of such a system have dimension 3. Finally if $\overline{\varphi}$ is neither an equilibrium point nor a periodic orbit, then the corresponding relative set is diffeomorphic to a cylinder $\mathbb{R} \times \mathbb{S}^1$. In particular, we have the following result.

THEOREM 4.1. *Let* $\overline{\varphi} = \{\varphi(t) = (r(t), \dot r(t), z(t), \dot z(t)) \ : \ t \in \mathbb{R}\}$ *be an orbit of the reduced isosceles problem* (3.1) *for a fixed value of* $c \neq 0$*; and let* $\gamma_{\varphi,\theta_0,c} = \{\gamma_{\varphi,\theta_0,c}(t) = (r(t), \dot r(t), z(t), \dot z(t), \theta(t), \dot\theta(t)) \ : \ t \in \mathbb{R}\}$ *be the orbit of the isosceles problem* (2.1) *with* $\theta(t) = F(t) + \theta_0$ *(see* (4.1)*) for a fixed* $\theta_0 \in \mathbb{S}^1$*. Then* $\mathcal{E}_{\varphi,c}$ *is diffeomorphic to one of the following manifolds:*

1. *A circle* $\mathbb{S}^1 \subset \mathcal{E}_c$ *formed by a periodic orbit of* (2.1) *with period* $128\pi c^3/(1+8\mu)^2$ *if* $\overline{\varphi} = (8c^2/(1+8\mu), 0, 0, 0)$ *is the equilibrium point of* (3.1)*. This periodic orbit is known as the collinear solution of Euler for the* 3*-body problem (for more details see* [21]*).*
2. *A two-dimensional torus* $\mathbb{S}^1 \times \mathbb{S}^1 \subset \mathcal{E}_c$ *if* $\overline{\varphi}$ *is a* T*-periodic orbit. Moreover this torus is formed by the union of*
   (a) *periodic orbits of period* mT *if* $F(T) = 2\pi l/m$ *with* $l \in \mathbb{Z}$, $m \in \mathbb{N}$ *and* l, m *coprime;*
   (b) *quasi-periodic orbits if* $F(T) = \omega 2\pi$ *with* ω *an irrational number.*
3. *A cylinder* $\mathbb{S}^1 \times \mathbb{R} \subset \mathcal{E}_c$ *if* $\overline{\varphi}$ *is neither the equilibrium point nor a periodic orbit.*

5. Symmetries. It is easy to check that the equations of motion of the reduced isosceles problem (3.1) are invariant under the symmetry

$$(t, r, \dot r, z, \dot z) \longmapsto (-t, r, -\dot r, -z, \dot z). \tag{5.1}$$

This means that if $\varphi(t) = (r(t), \dot r(t), z(t), \dot z(t))$ is a solution of system (3.1), then also $\psi(t) = (r(-t), -\dot r(-t), -z(-t), \dot z(-t))$ is a solution. We note that in the configuration space $\{(r, z) \in \mathbb{R}^+ \times \mathbb{R}\}$ this symmetry corresponds to a symmetry with respect to the r-axis, so in what follows it will be denoted by the *r-symmetry*. On the other hand, in the configuration space $\{(r, z, \theta) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{S}^1\}$ the r-symmetry would correspond to a symmetry with respect to the plane defined by the motion of the particles P_1 and P_2.

This symmetry can be used, in a standard way, to find periodic solutions as follows. Suppose that $\varphi(t)$ crosses orthogonally the r-axis at a time $t = 0$; that is, $z(0) = 0$ and $\dot r(0) = 0$. Using symmetry (5.1) we have that the two solutions $\varphi(t)$ and $\psi(t)$ coincide at $t = 0$; then by the theorem of uniqueness of solutions of an ordinary differential equation they must be the same. If there is another time such that the solution $\varphi(t)$ crosses the r-axis orthogonally, then by symmetry (5.1) the orbit of $\varphi(t)$ must be closed, and $\varphi(t)$ is called an *r-symmetric periodic solution*.

Since system (3.1) is autonomous, the origin of time can be chosen arbitrarily. Thus, if $\gamma(t)$ is a solution of (3.1) that crosses the r-axis in a point $\mathbf{p}$ at $t = t_0$, then $\varphi(t) = \gamma(t + t_0)$ is a solution of (3.1) that crosses the r-axis in the point $\mathbf{p}$ at $t = 0$. Therefore we have proved the following well-known result.

PROPOSITION 5.1. *Let* $\varphi(t) = (r(t), \dot r(t), z(t), \dot z(t))$ *be a solution of the reduced* *isosceles problem* (3.1)*. If $\dot r(t)$ and $z(t)$ are zero at $t = t_0$ and at $t = t_0 + T/2$ but are not simultaneously zero at any value of $t \in (t_0, t_0 + T/2)$, then $\varphi(t)$ is an r-symmetric periodic solution of period T.*

Equations (3.1) are also invariant under the symmetry

$$(t, r, \dot r, z, \dot z) \longmapsto (-t, r, -\dot r, z, -\dot z), \tag{5.2}$$

i.e., the time reversibility symmetry, which will be denoted in what follows by the *t-symmetry.* As in the r-symmetry we can introduce the notion of *t-symmetric periodic solutions*, which are characterized as follows.

PROPOSITION 5.2. *Let $\varphi(t) = (r(t), \dot r(t), z(t), \dot z(t))$ be a solution of the reduced isosceles problem* (3.1)*. If $\dot r(t)$ and $\dot z(t)$ are zero at $t = t_0$ and at $t = t_0 + T/2$ but are not simultaneously zero at any value of $t \in (t_0, t_0 + T/2)$, then $\varphi(t)$ is a t-symmetric periodic solution of period T.*

We note that there could be periodic solutions of (3.1) that are simultaneously r- and t-symmetric. These periodic solutions will be called *doubly symmetric periodic solutions* (see, for instance, [12] for more information about doubly symmetric periodic orbits) and are characterized by the following result.

PROPOSITION 5.3. *Let $\varphi(t) = (r(t), \dot r(t), z(t), \dot z(t))$ be a solution of the reduced isosceles problem* (3.1)*.*

1. *If $\dot r(t)$ and $z(t)$ are zero at $t = t_0$ and $\dot r(t)$ and $\dot z(t)$ are zero at $t = t_0 + T/4$ but are not simultaneously zero at any value of $t \in (t_0, t_0 + T/4)$, then $\varphi(t)$ is a doubly symmetric periodic solution of period T.*
2. *If $\dot r(t)$ and $\dot z(t)$ are zero at $t = t_0$ and $\dot r(t)$ and $z(t)$ are zero at $t = t_0 + T/4$ but are not simultaneously zero at any value of $t \in (t_0, t_0 + T/4)$, then $\varphi(t)$ is a doubly-symmetric periodic solution of period T.*

6. Restricted isosceles problems. To obtain the restricted isosceles problems we assume that the value of the mass m_3 is infinitesimally small (i.e., $\mu = 0$). Then the equations of motion of the restricted isosceles problem become

$$\ddot r = r\dot\theta^2 - \frac{1}{8\,r^2}, \qquad \ddot z = -\frac{z}{(z^2 + r^2)^{3/2}}, \qquad \frac{d}{dt}(r^2\dot\theta) = 0. \tag{6.1}$$

Notice that the first and the third equations of (6.1) do not depend on z; moreover they are the equations of motion of a 2-body problem in polar coordinates. This means that the particles P_1 and P_2 (the *primaries*) move on the plane $z = 0$ describing a solution of this 2-body problem. Moreover the particle P_3 that lies on the straight line orthogonal to the plane containing P_1 and P_2 that passes through their center of mass moves under the gravitational attraction of the previous two but does not influence their motion. Thus, for every solution $(r(t), \theta(t))$ of that 2-body problem, system (6.1) defines a different restricted isosceles problem; it can be a circular, elliptic, parabolic, hyperbolic, elliptic collision, parabolic collision, or hyperbolic collision restricted isosceles problem depending on the nature of the solution $(r(t), \theta(t))$.

As in the isosceles problem (2.1) if we assume that the value of the angular momentum is fixed at $C = c$ for some $c \neq 0$, then we can reduce the dimension of the phase space by two, obtaining the *reduced restricted isosceles problem*

$$\ddot r = \frac{c^2}{r^3} - \frac{1}{8\,r^2}\ , \qquad \ddot z = -\frac{z}{(z^2 + r^2)^{3/2}}. \tag{6.2}$$

In this work we are interested only in the periodic solutions of system (6.2) for $c \neq 0$. So, we will consider only the reduced circular and elliptic restricted isosceles problems, which we will call *reduced circular Sitnikov problem* and *reduced elliptic Sitnikov problem*, respectively.

7. On the circular restricted isosceles problem. Without loss of generality we can assume that the primaries describe a circular orbit of radius $1/2$ (or, equivalently, a circular orbit of period 2π). This corresponds to fixing the value of the angular momentum to $c = 1/4$. Then the equation of motion for the infinitesimal mass becomes

$$\ddot{z} = -\frac{z}{(z^2 + 1/4)^{3/2}}, \tag{7.1}$$

which is the equation of the known *circular Sitnikov problem*.

Assume that $(z(t), \dot{z}(t))$ is a solution of (7.1) with arbitrary initial conditions $z(0) = z_0$ and $\dot{z}(0) = \dot{z}_0$. Then it is clear that $\varphi(t) = (r(t) = 1/2, \dot{r}(0) = 0, z(t), \dot{z}(t))$ is a solution of the reduced circular Sitnikov problem

$$\ddot{r} = \frac{1}{16\, r^3} - \frac{1}{8\, r^2}, \qquad \ddot{z} = -\frac{z}{(z^2 + r^2)^{3/2}}, \tag{7.2}$$

with initial conditions $r(0) = 1/2$, $\dot{r}(0) = 0$, $z(0) = z_0$, $\dot{z}(0) = \dot{z}_0$. Next we analyze the periodic solutions of this problem.

Since we have taken $r(t) = 1/2$, $\dot{r}(t) = 0$, it's clear that $\varphi(t) = (r(t), \dot{r}(t), z(t), \dot{z}(t))$ is a periodic solution of the reduced circular Sitnikov problem with period T if and only if $(z(t), \dot{z}(t))$ is a periodic solution of the circular Sitnikov problem (7.1) with period T. So, we start summarizing the basic results about periodic solutions of the circular Sitnikov problem (7.1) that are needed for the development of this work. Then we will analyze the periodic solutions of the reduced circular Sitnikov problem and their relationship with the corresponding solutions of the circular restricted isosceles problem.

7.1. Periodic solutions of the circular Sitnikov problem. Equation (7.1) defines an integrable Hamiltonian system of one degree of freedom with Hamiltonian $H = v^2/2 - \left(z^2 + 1/4\right)^{-1/2}$, where $v = \dot{z}$. The orbits for the circular Sitnikov problem in the energy level h are described by the curve $H = h$, where h varies in $[-2, \infty)$.

The circular Sitnikov problem has been studied by several authors. In 1907 Pavanini [19] expressed its solutions by means of Weierstrassian elliptic functions. Four years later MacMillan [14] expressed the solutions in terms of Jacobian elliptic functions (a detailed description of this work can be found in Stumpff [22]). Some other analytical expressions for the solutions of this problem can be found, for instance, in [23], [2], and [24]. In particular, in this paper we will use the analytical expressions of the solutions of the circular Sitnikov problem for $h > -2$ that appear in [2], which are given in terms of Jacobian elliptic functions. A detailed description of all Jacobian elliptic functions to be used in this paper can be found in [4] and [8].

We remark that the knowledge of an analytic expression for the solutions of the circular Sitnikov problem plays a key role in our analysis, because it allows us to prove our results analytically.

In what follows we use the following notation for the Jacobian elliptic functions: $\operatorname{sn}\nu = \operatorname{sn}(\nu, k)$, $\operatorname{cn}\nu = \operatorname{cn}(\nu, k)$, $\operatorname{dn}\nu = \operatorname{dn}(\nu, k)$ are the *sine*, *cosine*, and *delta amplitude* Jacobian elliptic functions, respectively; $F(\nu) = F(\operatorname{am}(\nu), k)$, $E(\nu) = E(\operatorname{am}(\nu), k)$, $\Pi(\nu, 2k^2) = \Pi(\operatorname{am}(\nu), 2k^2, k)$ are the *normal elliptic integral of the first*, *second*, and *third kind*, respectively; $\operatorname{am}(\nu)$ is the *amplitude* Jacobian elliptic function;

and finally $K = K(k) = F(\pi/2, k)$, $E = E(k) = E(\pi/2, k)$, $\Pi(\alpha^2, k) = \Pi(\pi/2, \alpha^2, k)$ are the *complete elliptic integrals of the first*, *second*, and *third kind*, respectively (see [4] or [8] for the precise definitions).

Using the analytic expression for the solutions of the circular Sitnikov problem given in Theorem A of [2], we see that the periodic solutions of that problem can be written as follows (see [8] for more details).

LEMMA 7.1. *The periodic solutions of the circular Sitnikov problem have energy $-2 < h < 0$ and can be written as*

$$(z(t), \dot z(t)) = \left(\frac{k \operatorname{sn} \nu \operatorname{dn} \nu}{1 - 2k^2 \operatorname{sn}^2 \nu} \ , \ 2\sqrt{2} k \operatorname{cn} \nu \right), \tag{7.3}$$

where $k = \sqrt{2+h}/2$ and ν is the function of t defined implicitly by

$$\begin{aligned} t &= \frac{\sqrt{2}}{8(1-2k^2)} \left[2E(\nu) - \nu + \Pi(\nu, 2k^2) - 4k^2 \frac{\operatorname{sn}\nu \operatorname{cn}\nu \operatorname{dn}\nu}{1 - 2k^2 \operatorname{sn}^2\nu} \right] + C \\ &= \tau(\nu, k) + C. \end{aligned}$$

Here C is an integration constant whose value depends on the initial conditions of the periodic solution $(z(t), \dot z(t))$.

Since $\operatorname{sn}\nu$ and $\operatorname{cn}\nu$ are periodic functions of period $4K$ and $\operatorname{dn}\nu$ is a periodic function of period $2K$ (see formulas 122 in [4]), from (7.3) we see that the period in the new time ν is $4K$, where $K = K(k)$ is the complete elliptic integral of the first kind and $k = \sqrt{2+h}/2$. Moreover the period in the real time t is given by

$$T = \frac{\sqrt{2}}{2(1-2k^2)} [2E(k) - K(k) + \Pi(2k^2, k)]; \tag{7.4}$$

for more details see Theorem 2.3 in [8].

We note that (7.1) is autonomous, so the origin of time can be chosen arbitrarily. In particular, in this paper we are interested only in periodic solutions $(z(t), \dot z(t))$ having initial conditions either $z(0) = 0$ or $\dot z(0) = 0$. The following lemma, taken from [8], gives the values of the integration constant C for those initial conditions.

LEMMA 7.2. *Let T be the period of the periodic solution $(z(t), \dot z(t))$ given in (7.4).*

1. *If $(z(t), \dot z(t))$ has initial conditions $z(0) = 0$ and $\dot z(0) = \sqrt{2h+4}$, then taking $\nu(0) = 0$, we have $t = \tau(\nu, k)$.*
2. *If $(z(t), \dot z(t))$ has initial conditions $z(0) = 0$ and $\dot z(0) = -\sqrt{2h+4}$, then taking $\nu(0) = 2K$, we have $t = \tau(\nu, k) - T/2$.*
3. *If $(z(t), \dot z(t))$ has initial conditions $z(0) = \sqrt{\frac{1}{h^2} - \frac{1}{4}}$ and $\dot z(0) = 0$, then taking $\nu(0) = K$, we have $t = \tau(\nu, k) - T/4$.*
4. *If $(z(t), \dot z(t))$ has initial conditions $z(0) = -\sqrt{\frac{1}{h^2} - \frac{1}{4}}$ and $\dot z(0) = 0$, then taking $\nu(0) = 3K$, we have $t = \tau(\nu, k) - 3T/4$.*

In order to simplify computations we will usually work with the new time ν instead of the real time t, but always keeping in mind that ν is a function of t via Lemma 7.2. The two following lemmas taken also from [8] give some relationships between the real time t and the new time ν that will be useful later on.

LEMMA 7.3. *Let T be the period of the periodic solution $(z(t), \dot z(t))$.*

1. *$\nu(t + qT) = \nu(t) + q4K$ for all $t \in \mathbb{R}$ and for all $q \in \mathbb{N}$.*
2. *$\nu(t + qT/2) = \nu(t) + q2K$ for all $t \in \mathbb{R}$ and for all $q \in \mathbb{N}$.*

LEMMA 7.4. *Let T be the period of the solution $(z(t), \dot{z}(t))$. If $\nu(0) = lK$ for $l = 0, 1, 2, 3$, then $\nu(qT/4) = (l + q)K$ for all $q \in \mathbb{N}$.*

The following result gives the properties of the function $T = T(h)$.

THEOREM 7.5. *The period T satisfies*

1. $\lim_{h\to -2} T(h) = \pi/\sqrt{2}$;
2. $\lim_{h\to 0} T(h) = \infty$;
3. $dT/dh > 0$ *for all* $h \in (-2, 0)$;
4. $\lim_{h\to -2} dT/dh = \pi(1 + 4\sqrt{2})/16$;
5. $\lim_{h\to 0} dT/dh = \infty$.

Proof. See the proof of Theorem C in [2]. ∎

Theorem 7.5 assures the existence of periodic orbits of the circular Sitnikov problem with period $T = T(h)$ for all $T > \pi/\sqrt{2}$. In fact, since $T = T(h)$ is an injective function there is a one-to-one correspondence between $h \in (-2, 0)$ and $T \in (\pi/\sqrt{2}, \infty)$, so we can characterize the periodic orbits either by the period or by the energy.

7.2. Periodic solutions of the reduced circular Sitnikov problem. Notice that equations (7.2) are invariant under symmetries (5.1) and (5.2). These symmetries can be used to obtain symmetric periodic solutions for the reduced circular Sitnikov problem. It is not difficult to prove the next result.

PROPOSITION 7.6. *All periodic orbits of the reduced circular Sitnikov problem are doubly symmetric periodic orbits.*

We note that the periodic solutions of the reduced circular Sitnikov problem are periodic solutions for the infinitesimal mass, but in general they are not periodic solutions involving the three masses; that is, they are not periodic solutions of the circular restricted isosceles problem. Since the primaries describe a circular solution of a 2-body problem with period 2π, the only periodic orbits of the circular Sitnikov problem that give periodic orbits involving the three masses are the ones that have a period commensurable with 2π; that is, $T = T(h) = 2\pi p/q$ for some $p, q \in \mathbb{N}$ coprime. In this case the period of the corresponding orbit involving the three masses is $\tau = 2\pi p = qT(h)$. That is, during a period τ, the primaries have completed p revolutions and the infinitesimal mass has completed q revolutions.

7.3. Invariant tori of the circular restricted isosceles problem. From section 7.2, we have the following result, which can be obtained easily from Theorem 4.1.

PROPOSITION 7.7. *Let $\{(z_h(t), \dot{z}_h(t)) : t \in \mathbb{R}\}$ be a periodic orbit of the circular Sitnikov problem with energy h for some $h \in (-2, 0)$; and let $\overline{\varphi}_h = \{\varphi_h(t) = (r(t) = 1/2, \dot{r}(t) = 0, z_h(t), \dot{z}_h(t)) : t \in \mathbb{R}\}$ be its corresponding orbit of the reduced circular Sitnikov problem. Then the relative set of the circular restricted isosceles problem associated to the orbit $\overline{\varphi}_h$ is diffeomorphic to a two-dimensional torus $\mathbb{S}^1 \times \mathbb{S}^1$. Moreover, this relative torus is formed by the union of*

1. *periodic orbits of period qT if $T = T(h) = 2\pi p/q$ for some $p, q \in \mathbb{N}$ coprime and $p > q/(2\sqrt{2})$;*
2. *quasi-periodic orbits if $T = T(h) = 2\pi\omega$ for some irrational $\omega > 1/(2\sqrt{2})$.*

8. On the elliptic restricted isosceles problem. We assume that the primaries are describing an elliptic orbit of the 2-body problem with period 2π and eccentricity e. This corresponds to fixing the value of the angular momentum to $c = c_e = \sqrt{1 - e^2}/4$. Then, choosing conveniently the origin of time, a solution of the reduced elliptic Sitnikov problem is a solution $\varphi(t) = (r(t), \dot{r}(t), z(t), \dot{z}(t))$ of

$$\ddot{r} = \frac{1 - e^2}{16\, r^3} - \frac{1}{8\, r^2}\ , \qquad \ddot{z} = -\frac{z}{(z^2 + r^2)^{3/2}}, \tag{8.1}$$

with initial conditions $r(0) = (1 \pm e)/2$, $\dot r(0) = 0$, $z(0) = z_0$, $\dot z(0) = \dot z_0$ for some $z_0, \dot z_0 \in \mathbb{R}$.

Since $r(t)$ is a 2π-periodic function, the periodic solutions of the reduced elliptic Sitnikov problem must have period that is a multiple of 2π. Moreover $\varphi(t) = (r(t), \dot r(t), z(t), \dot z(t))$ is a periodic solution of the reduced elliptic Sitnikov problem with period $T = 2k\pi$ for some $k \in \mathbb{N}$ if and only if $(z(t), \dot z(t))$ is a periodic solution with period $T = 2k\pi$ of the elliptic Sitnikov problem

$$\ddot z = -\frac{z}{(z^2 + r(t)^2)^{3/2}}.$$

It is clear that equations (8.1) are invariant under symmetries (5.1) and (5.2). These symmetries can be used to obtain symmetric periodic solutions for the reduced elliptic Sitnikov problem. We remark that symmetries (5.1) and (5.2) for the reduced elliptic Sitnikov problem correspond to the r- and the t-symmetry of the elliptic Sitnikov problem defined in [7] and [8].

8.1. Periodic solutions of the reduced elliptic Sitnikov problem. In section 7.2 we have seen that all periodic orbits of the reduced circular Sitnikov problem are doubly symmetric periodic orbits. This fact does not occur when we consider the reduced elliptic Sitnikov problem, as follows from the next result.

PROPOSITION 8.1. *For the reduced elliptic Sitnikov problem there exist four different types of periodic orbits: nonsymmetric periodic orbits, doubly symmetric periodic orbits, and r- and t-symmetric periodic orbits that are not doubly symmetric.*

Proof. See the proof of Propositions 12, 15, and 23 in [7]. ∎

On the other hand, [8] gives initial conditions for some symmetric periodic solutions of the elliptic Sitnikov problem (or, equivalently, the reduced elliptic Sitnikov problem) with sufficiently small values of the eccentricity $e > 0$. These initial conditions are obtained from the analytic continuation of the known periodic solutions of the reduced circular Sitnikov problem to symmetric periodic solutions of the reduced elliptic Sitnikov problem for sufficiently small values of the eccentricity e. Later on, in section 11, the symmetric periodic solutions of the reduced elliptic Sitnikov problem given in [8] will be continued to the reduced isosceles problem for sufficiently small values of $\mu > 0$. Here we summarize the main results of [8] about symmetric periodic orbits of the reduced elliptic Sitnikov problem.

In what follows $\varphi_c(t; \mathbf{x}_0, \mu) = (r(t; \mathbf{x}_0, \mu), \dot r(t; \mathbf{x}_0, \mu), z(t; \mathbf{x}_0, \mu), \dot z(t; \mathbf{x}_0, \mu))$, with $\mathbf{x}_0 = (r_0, \dot r_0, z_0, \dot z_0)$, denotes the solution of the reduced isosceles problem (3.1) with angular momentum $C = c \neq 0$, satisfying the initial conditions $r(0; r_0, \dot r_0, z_0, \dot z_0, \mu) = r_0$, $\dot r(0; r_0, \dot r_0, z_0, \dot z_0, \mu) = \dot r_0$, $z(0; r_0, \dot r_0, z_0, \dot z_0, \mu) = z_0$, $\dot z(0; r_0, \dot r_0, z_0, \dot z_0, \mu) = \dot z_0$.

THEOREM 8.2. *Given $p, q \in \mathbb{N}$ coprime with $p > q/(2\sqrt{2})$, let $\varphi_{1/4}\big(t; r_0 = 1/2, \dot r_0 = 0, z_0 = 0, \dot z_0 = \dot z_0^* = \pm\sqrt{2h+4}, \mu = 0\big)$ be a periodic solution of the reduced circular Sitnikov problem with period $T = 2\pi p/q$.*

1. *This solution can be continued to two families $\varphi_{c_e}(t; r_0 = (1-e)/2, \dot r_0 = 0, z_0 = 0, \dot z_0 = \dot z_0^P = \dot z_0^* + O(e), \mu = 0)$ and $\varphi_{c_e}(t; r_0 = (1+e)/2, \dot r_0 = 0, z_0 = 0, \dot z_0 = \dot z_0^A = \dot z_0^* + O(e), \mu = 0)$ of r-symmetric periodic solutions of the reduced elliptic Sitnikov problem having period $\tau = 2\pi p = qT$ for $e > 0$ sufficiently small.*
2. *If p is odd, then those r-symmetric periodic solutions are not doubly symmetric, whereas if p is even, then they are doubly symmetric.*

Proof. See the proof of Theorem 4.4 in [8]. ∎

THEOREM 8.3. *Given $p, q \in \mathbb{N}$ coprime with $p > q/(2\sqrt{2})$, let $\varphi_{1/4}(t; r_0 = 1/2,$*

$\dot{r}_0 = 0, z_0 = z_0^* = \pm\sqrt{\frac{1}{h^2} - \frac{1}{4}}, \dot{z}_0 = 0, \mu = 0)$ *be a periodic solution of the reduced circular Sitnikov problem with period* $T = 2\pi p/q$.

1. *This solution can be continued to two families* $\varphi_{c_e}(t; r_0 = (1-e)/2, \dot{r}_0 = 0, z_0 = z_0^P = z_0^* + O(e), \dot{z}_0 = 0, \mu = 0)$ *and* $\varphi_{c_e}(t; r_0 = (1+e)/2, \dot{r}_0 = 0, z_0 = z_0^A = z_0^* + O(e), \dot{z}_0 = 0, \mu = 0)$ *of t-symmetric periodic solutions of the reduced elliptic Sitnikov problem having period* $\tau = 2\pi p = qT$ *for* $e > 0$ *sufficiently small.*
2. *If p is odd, then those t-symmetric periodic solutions are not doubly symmetric, whereas if p is even, then they are doubly symmetric.*

Proof. See the proof of Theorem 4.6 in [8]. □

We note that in Theorems 8.2 and 8.3 we continue four different initial conditions of the periodic orbit of the reduced circular Sitnikov problem with period $T = 2\pi p/q$ for given $p, q \in \mathbb{N}$ coprime, $p > q/(2\sqrt{2})$; they are $\varphi_{1/4}\left(t; 1/2, 0, 0, \sqrt{2h+4}, 0\right)$ and $\varphi_{1/4}\left(t; 1/2, 0, 0, -\sqrt{2h+4}, 0\right)$ in Theorem 8.2, and $\varphi_{1/4}(t; 1/2, 0, \sqrt{\frac{1}{h^2} - \frac{1}{4}}, 0, 0)$ and $\varphi_{1/4}(t; 1/2, 0, -\sqrt{\frac{1}{h^2} - \frac{1}{4}}, 0, 0)$ in Theorem 8.3. These four initial conditions are continued to eight families of periodic orbits of the reduced elliptic Sitnikov problem for $e > 0$ sufficiently small. The following theorem says how many of these eight families of periodic orbits are really different (see [8] for more details).

THEOREM 8.4. *The periodic solutions of the reduced circular Sitnikov problem with period* $T = 2\pi p/q$*, for given* $p, q \in \mathbb{N}$ *coprime* $p > q/(2\sqrt{2})$*, can be continued to*

1. *two families of r-symmetric periodic orbits and two families of t-symmetric periodic orbits (that are not doubly symmetric) of the reduced elliptic Sitnikov problem with period* $\tau = 2\pi p = qT$*, for* $e > 0$ *sufficiently small, when p is odd;*
2. *two families of doubly symmetric periodic orbits of the reduced elliptic Sitnikov problem with period* $\tau = 2\pi p = qT$*, for* $e > 0$ *sufficiently small, when p is even.*

Proof. See the proof of Theorem 4.15 in [8]. □

8.2. Invariant tori of the elliptic restricted isosceles problem. From Theorem 4.1(2)(a), the next result follows.

PROPOSITION 8.5. *Let* $\overline{\varphi} = \{\varphi(t) = (r(t), \dot{r}(t), z(t), \dot{z}(t)) : t \in \mathbb{R}\}$ *be a periodic orbit of the reduced elliptic Sitnikov problem with period* $\tau = 2\pi n$ *for some* $n \in \mathbb{N}$*. Then the relative set of the restricted isosceles problem associated to the orbit* $\overline{\varphi}$ *is diffeomorphic to a two-dimensional torus* $\mathbb{S}^1 \times \mathbb{S}^1 \subset \mathcal{E}_{c_e}$*, which is formed by periodic orbits of period* τ.

We remark that the orbits of the circular restricted isosceles problem coming from periodic orbits of the reduced circular Sitnikov problem are not in general periodic orbits (see Proposition 7.7).

By means of Propositions 7.7 and 8.5, Theorem 8.4 can be extended to the restricted isosceles problem, obtaining the following result.

THEOREM 8.6. *Let* Γ_{pq} *be the periodic two-dimensional tori of the circular restricted isosceles problem that comes from the periodic orbit of the reduced circular Sitnikov problem with period* $T = p2\pi/q$*,* $p, q \in \mathbb{N}$ *coprime and* $p > q/2\sqrt{2}$*. Then* Γ_{pq} *can be continued to two or four families (two for even p and four for odd p) of periodic two-dimensional tori of the elliptic restricted isosceles problem.*

9. Variational equations. The main objective of this work is to continue the known symmetric periodic orbits of the reduced circular and elliptic Sitnikov problems to symmetric periodic orbits of the reduced isosceles problem for $\mu > 0$ sufficiently small. Those periodic orbits will be continued by using the classical analytic continuation method of Poincaré (for details see [21] or [15]). In order to apply this method to our problem we must know the solution of the variational equations of the reduced circular and elliptic Sitnikov problems along the periodic solutions that we want to continue. In this section we will analyze those variational equations.

Let $(r(t), R(t), z(t), Z(t))$ be a solution of the reduced circular ($e = 0$) or elliptic ($0 < e < 1$) Sitnikov problem

$$\dot{r} = R, \qquad \dot{R} = \frac{1-e^2}{16\,r^3} - \frac{1}{8\,r^2}, \qquad \dot{z} = Z, \qquad \dot{Z} = -\frac{z}{(z^2+r^2)^{3/2}}, \tag{9.1}$$

with initial conditions $r(0) = r_0 = (1 \pm e)/2$, $R(0) = R_0 = 0$, $z(0) = z_0$, $Z(0) = Z_0$. In particular, $(r(t), R(t))$ is a circular or elliptic solution of the Kepler problem

$$\dot{r} = R\ , \qquad \dot{R} = \frac{1-e^2}{16\,r^3} - \frac{1}{8\,r^2}; \tag{9.2}$$

and $(z(t), Z(t))$ is a solution of the circular or elliptic Sitnikov problem

$$\dot{z} = Z\ , \qquad \dot{Z} = -\frac{z}{(z^2+r^2(t))^{3/2}} \tag{9.3}$$

(see sections 7 and 8).

The variational equations of system (9.1) along the solution curve $(r(t), R(t), z(t), Z(t))$ are given by the matrix differential equation

$$\frac{d}{dt}A = B(t)\,A, \tag{9.4}$$

with initial condition $A(0) = I$ (the 4×4 identity matrix), where

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \qquad B(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ b_1(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ b_2(t) & 0 & b_3(t) & 0 \end{pmatrix},$$

with A_1, A_2, A_3, and A_4 given by

$$\begin{pmatrix} \dfrac{\partial r}{\partial r_0} & \dfrac{\partial r}{\partial R_0} \\ \dfrac{\partial R}{\partial r_0} & \dfrac{\partial R}{\partial R_0} \end{pmatrix}, \quad \begin{pmatrix} \dfrac{\partial r}{\partial z_0} & \dfrac{\partial r}{\partial Z_0} \\ \dfrac{\partial R}{\partial z_0} & \dfrac{\partial R}{\partial Z_0} \end{pmatrix}, \quad \begin{pmatrix} \dfrac{\partial z}{\partial r_0} & \dfrac{\partial z}{\partial R_0} \\ \dfrac{\partial Z}{\partial r_0} & \dfrac{\partial Z}{\partial R_0} \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} \dfrac{\partial z}{\partial z_0} & \dfrac{\partial z}{\partial Z_0} \\ \dfrac{\partial Z}{\partial z_0} & \dfrac{\partial Z}{\partial Z_0} \end{pmatrix},$$

respectively, and

$$b_1(t) = -\frac{3(1-e^2)}{16\,r^4(t)} + \frac{1}{4\,r^3(t)}, \quad b_2(t) = \frac{3\,r(t)\,z(t)}{(z^2(t)+r^2(t))^{5/2}}, \quad b_3(t) = \frac{2\,z^2(t) - r^2(t)}{(z^2(t)+r^2(t))^{5/2}}.$$

If we denote $q_1 = r_0$, $q_2 = R_0$, $q_3 = z_0$, and $q_4 = Z_0$ system (9.4) can be written like the linear system of differential equations,

$$\left\{ \begin{array}{l} \dfrac{d}{dt}\left(\dfrac{\partial r}{\partial q_i}\right) = \dfrac{\partial R}{\partial q_i}\ , \\[2ex] \dfrac{d}{dt}\left(\dfrac{\partial R}{\partial q_i}\right) = \left(-\dfrac{3(1-e^2)}{16\,r^4(t)} + \dfrac{1}{4\,r^3(t)}\right)\dfrac{\partial r}{\partial q_i}\ , \end{array} \right. \tag{9.5}$$

$$\left\{\begin{aligned} &\frac{d}{dt}\left(\frac{\partial z}{\partial q_i}\right) = \frac{\partial Z}{\partial q_i}\ , \\ &\frac{d}{dt}\left(\frac{\partial Z}{\partial q_i}\right) = \frac{3\, r(t) z(t)}{(z^2(t)+r^2(t))^{5/2}}\frac{\partial r}{\partial q_i} + \frac{2\, z^2(t) - r^2(t)}{(z^2(t)+r^2(t))^{5/2}}\frac{\partial z}{\partial q_i}\ , \end{aligned}\right. \tag{9.6}$$

with initial conditions

$$\frac{\partial r}{\partial q_i}(0) = \delta_{1,i}\ , \qquad \frac{\partial R}{\partial q_i}(0) = \delta_{2,i}\ , \qquad \frac{\partial z}{\partial q_i}(0) = \delta_{3,i}\ , \qquad \frac{\partial Z}{\partial q_i}(0) = \delta_{4,i}\ ,$$

where $i = 1, \dots, 4$, $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$.

Since equations (9.5) do not depend on $\partial z/\partial q_i$ and $\partial Z/\partial q_i$, they can be solved separately. Thus, the derivatives $\partial r/\partial r_0$, $\partial r/\partial R_0$, $\partial R/\partial r_0$, and $\partial R/\partial R_0$ are given by the solution of the matrix differential equation

$$\frac{d}{dt}\ A_1 = \begin{pmatrix} 0 & 1 \\ -\dfrac{3(1-e^2)}{16\, r^4(t)} + \dfrac{1}{4\, r^3(t)} & 0 \end{pmatrix} A_1\ , \tag{9.7}$$

with initial condition $A_1(0) = I$ (the 2×2 identity matrix); that is, they are given by the solution of the variational equations of the Kepler problem (9.2) along the solution curve $(r(t), R(t))$.

On the other hand,

$$\frac{\partial r}{\partial z_0}(t) = 0\ , \qquad \frac{\partial R}{\partial z_0}(t) = 0\ , \qquad \frac{\partial r}{\partial Z_0}(t) = 0\ , \qquad \frac{\partial R}{\partial Z_0}(t) = 0, \tag{9.8}$$

because the first two equations of (9.1) do not depend on z and Z; consequently changes on the initial conditions z_0 and Z_0 do not affect the solution $(r(t), R(t))$.

By (9.6) and (9.8), the derivatives $\partial z/\partial z_0$,$\partial z/\partial Z_0$,$\partial Z/\partial z_0$, and $\partial Z/\partial Z_0$ are given by the solution of the matrix differential equation

$$\frac{d}{dt}\ A_4 = \begin{pmatrix} 0 & 1 \\ -\dfrac{2\, z^2(t) - r^2(t)}{(z^2(t)+r^2(t))^{5/2}} & 0 \end{pmatrix} A_4, \tag{9.9}$$

with initial condition $A_4(0) = I$ (the 2×2 identity matrix); that is, they are given by the solution of the variational equations of the circular or elliptic Sitnikov problem (9.3) along the solution curve $(z(t), Z(t))$.

We note that we do not know an exact expression for the symmetric periodic solutions of the nonautonomous elliptic Sitnikov problem, and thus their variational equations cannot be solved explicitly. However, since the eccentricity e is sufficiently small, the solution of these variational equations may be expressed as a power series of the eccentricity e. We have computed analytically the terms of zero order in e. They are given by the variational equations of the circular Sitnikov problem.

Finally the derivatives $\partial z/\partial r_0$, $\partial z/\partial R_0$, $\partial Z/\partial r_0$, and $\partial Z/\partial R_0$ are obtained by solving the nonhomogeneous linear system of differential equations that comes from replacing in (9.6) $\partial r/\partial q_i$ and $\partial R/\partial q_i$, $i = 1, 2$, by the solutions $(\partial r/\partial q_i)(t)$ and $(\partial R/\partial q_i)(t)$ of the variational equations of the Kepler problem (9.2) along the solution curve $(r(t), R(t))$. If we know a fundamental matrix $\Phi(t)$ of the variational equations of the circular or elliptic Sitnikov problem (9.3) along the solution curve $(z(t), Z(t))$ (i.e., a fundamental matrix of the homogeneous system), then we can solve the nonhomogeneous one using the method of variation of constants (see, for instance, [11, p. 81]). Thus, for $i = 1, 2$, we have that

$$\begin{pmatrix} \dfrac{\partial z}{\partial q_i}(t) \\ \dfrac{\partial Z}{\partial q_i}(t) \end{pmatrix} = \Phi(t) \int_0^t \Phi^{-1}(s) \begin{pmatrix} 0 \\ \dfrac{3\, r(s) z(s)}{(z^2(s)+r^2(s))^{5/2}} \dfrac{\partial r}{\partial q_i}(s) \end{pmatrix} ds.$$

In order to compute the solution of the variational equations of the Kepler problem (9.2), for $0 \leqslant e < 1$, and of the circular Sitnikov problem (9.3) with $r(t) = 1/2$, we could use a theorem of Diliberto [10] on the integration of the homogeneous variational equations of a plane autonomous differential system in terms of geometric quantities along a given solution curve of the system (see also the paper of Chicone [5], where, in addition to using the Diliberto theorem to address his problem, he corrects a flaw in the theorem). But we compute here the solution of those variational equations directly using a result that appears in [8].

We note that the Kepler problem (9.2) and the circular Sitnikov problem (9.3) with $r(t) = 1/2$ can be written like a second order differential equation of the form

$$\ddot{x} = f(x). \tag{9.10}$$

The solution of the variational equations of (9.10) along a given nonconstant solution curve $x(t)$ are given by the following result.

Proposition 9.1. *The linear variational equations of* (9.10) *along a nonconstant solution curve $x(t)$ have a fundamental matrix $\Phi(t)$, satisfying that* $\det(\Phi(0)) = 1$, *which is given by*

$$\Phi(t) = \begin{pmatrix} \dot{x}(t) & g(t) \\ f(x(t)) & \dot{g}(t) \end{pmatrix},$$

where $g(t) = \dot{x}(t) \int \frac{dt}{\dot{x}^2(t)}$ without the constant due to integration.

Moreover, the solution of these variational equations is given by

$$\begin{pmatrix} \dfrac{\partial x}{\partial x_0}(t) & \dfrac{\partial x}{\partial y_0}(t) \\ \dfrac{\partial y}{\partial x_0}(t) & \dfrac{\partial y}{\partial y_0}(t) \end{pmatrix} = \Phi(t)\Phi^{-1}(0),$$

where $y = \dot{x}$, $x_0 = x(0)$, and $y_0 = \dot{x}(0)$.

Proof. See the proof of Proposition B.1 in [8]. ∎

9.1. Variational equations of the Kepler problem. We start computing a fundamental matrix of the variational equations (9.7) of the Kepler problem (9.2) for $0 \leqslant e < 1$ along an arbitrary elliptic solution (a circular solution if $e = 0$)

$$r(t) = \tfrac{1}{2}(1 - e\cos u). \tag{9.11}$$

As usual u is the eccentric anomaly which is a function of t via the Kepler's equation

$$u - e\sin u = t - \tau = M, \tag{9.12}$$

where M is the mean anomaly and τ is the time of pericenter passage. Later on we will give the solution of those variational equations when $(r(t), R(t))$ is the solution with initial conditions $r(0) = (1 \pm e)/2$ and $R = \dot{r}(0) = 0$.

We note that when $e = 0$ we cannot apply Proposition 9.1 to solve the variational equations (9.7) of the Kepler problem (9.2) along the solution curve (9.11), because $r(t) = 1/2$ is constant.

PROPOSITION 9.2. *When $e = 0$, the solution of the variational equations* (9.7) *of the Kepler problem* (9.2) *along the solution curve* (9.11) *is given by*

$$A_1(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Proof. The proof follows easily, noting that the solution of the variational equations (9.7) when $e = 0$ is a matrix whose columns are the solutions of the differential equation

$$\frac{d}{dt}\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix},$$

with initial conditions $\omega_1(0) = 1$, $\omega_2(0) = 0$ and $\omega_1(0) = 0$, $\omega_2(0) = 1$, respectively. ∎

When $0 < e < 1$, to solve the variational equations (9.7) of the Kepler problem (9.2) along the solution curve (9.11), we apply Proposition 9.1. Thus a fundamental matrix of those variational equations is given by

$$\Phi(t) = \begin{pmatrix} \Phi_{11}(t) & \Phi_{12}(t) \\ \Phi_{21}(t) & \Phi_{22}(t) \end{pmatrix} = \begin{pmatrix} \dot r(t) & g(t) \\ \dfrac{1-e^2}{16\,r^3(t)} - \dfrac{1}{8\,r^2(t)} & \dot g(t) \end{pmatrix},$$

where $g(t) = \dot r(t)\int \frac{dt}{\dot r^2(t)}$.

In order to simplify our computations we will work with the eccentric anomaly, u, instead of the real time, t, but keeping in mind that u is a function of t via (9.12) when it is necessary.

Replacing $r(t)$ by (9.11) in $\Phi_{21}(t)$ and simplifying we get that

$$\Phi_{21}(t) = -\frac{e^2 - e\cos u}{2(1-e\cos u)^3}\ . \tag{9.13}$$

Differentiating (9.11) with respect to t we obtain

$$\Phi_{11}(t) = \dot r(t) = \frac{dr}{du}\frac{du}{dt} = \frac{e\sin u}{2(1-e\cos u)}. \tag{9.14}$$

Substituting $\dot r(t)$ into $g(t)$ and working with the variable u instead of the variable t, we have that

$$\begin{aligned}\Phi_{12}(t) = g(t) &= \frac{2\sin u}{e(1-e\cos u)}\int \frac{(1-e\cos u)^3}{\sin^2 u}\,du \\ &= \frac{2}{e(1-e\cos u)}[-(1+3e^2)\cos u - 3e^2 u\sin u + e^3\sin^2 u + 3e + e^3].\end{aligned} \tag{9.15}$$

Finally, differentiating $g(t)$ we obtain

$$\begin{aligned}\Phi_{22}(t) = \dot g(t) = \frac{dg}{du}\frac{du}{dt} &= \frac{2}{e(1-e\cos u)^3}[\sin u(1-3e^2-3e^4) - 3e^2 u\cos u \\ &\quad + 5e^3\sin u\cos u + 3e^3 u + e^4\sin^3 u].\end{aligned} \tag{9.16}$$

In short, we have proved the following result.

PROPOSITION 9.3. *A fundamental matrix of the variational equations* (9.7) *of the Kepler problem* (9.2) *along the solution curve* (9.11), *when* $0 < e < 1$, *is* $\Phi(t) = (\Phi_{ij}(t))$, *where* $\Phi_{ij}(t)$, *with* $i, j = 1, 2$, *are given by* (9.13), (9.14), (9.15), *and* (9.16), *and* u *is the eccentric anomaly as a function of* t *via the Kepler equation* (9.12). *Moreover the solution of these variational equations is*

$$A_1(t) = \Phi(t)\Phi^{-1}(0). \tag{9.17}$$

Now we compute the solution of the variational equations (9.7) of the Kepler problem (9.2) along the elliptic solution $(r(t), R(t))$ with initial conditions $r(0) = (1 \pm e)/2$ and $R(0) = 0$.

Case $r(0) = (1 - e)/2$. Without loss of generality, we can assume that $u(0) = 0$. Then the Kepler equation (9.12) becomes

$$u - e\sin u = t. \tag{9.18}$$

Therefore, by Proposition 9.3, the fundamental matrix $\Phi(t)$ evaluated at $t = 0$ (or, equivalently, at $u = 0$) is given by $\Phi_{11}(0) = \Phi_{22}(0) = 0$, $\Phi_{12}(0) = -2(1 - e)^2/e$, $\Phi_{21}(0) = e/(2(1-e)^2)$. Therefore, from (9.17) after doing some computations, we get

$$\begin{aligned}
\frac{\partial r}{\partial r_0}(t) &= \frac{(1 + 3e^2)\cos u + 3e^2 u\sin u - e^3\sin^2 u - 3e - e^3}{(1 - e)^2(1 - e\cos u)},\\
\frac{\partial r}{\partial R_0}(t) &= (1 - e)^2\frac{\sin u}{(1 - e\cos u)},\\
\frac{\partial R}{\partial r_0}(t) &= -\frac{1}{(1 - e)^2(1 - e\cos u)^3}[(1 - 3e^2 - 3e^4)\sin u\\
&\quad - 3e^2 u\cos u + 5e^3\sin u\cos u + 3e^3 u + e^4\sin^3 u],\\
\frac{\partial R}{\partial R_0}(t) &= (1 - e)^2\frac{(\cos u - e)}{(1 - e\cos u)^3},
\end{aligned} \tag{9.19}$$

and u is the eccentric anomaly as a function of time via (9.18).

Case $r(0) = (1 + e)/2$. Without loss of generality, we can assume that $u(0) = \pi$, and the Kepler equation (9.12) becomes

$$u - e\sin u = t + \pi. \tag{9.20}$$

By Proposition 9.3, the fundamental matrix $\Phi(t)$ evaluated at $t = 0$, or, equivalently, at $u = \pi$, is given by $\Phi_{11}(0) = 0$, $\Phi_{12}(0) = 2(1+e)^2/e$, $\Phi_{21}(0) = -e/(2(1+e)^2)$, $\Phi_{22}(0) = 6e\pi/(1 + e)^2$. Thus, by (9.17) we have

$$\begin{aligned}
\frac{\partial r}{\partial r_0}(t) &= -\frac{(1 + 3e^2)\cos u + 3e^2 u\sin u - e^3\sin^2 u - 3e - e^3 - 3e^2\pi\sin u}{(1 + e)^2(1 - e\cos u)},\\
\frac{\partial r}{\partial R_0}(t) &= -(1 + e)^2\frac{\sin u}{1 - e\cos u},\\
\frac{\partial R}{\partial r_0}(t) &= \frac{1}{(1 + e)^2(1 - e\cos u)^3}[(1 - 3e^2 - 3e^4)\sin u - 3e^2 u\cos u\\
&\quad + 5e^3\sin u\cos u + 3e^3 u + e^4\sin^3 u + 3e^2\pi\cos u - 3e^3\pi],\\
\frac{\partial R}{\partial R_0}(t) &= -(1 + e)^2\frac{(\cos u - e)}{(1 - e\cos u)^3},
\end{aligned} \tag{9.21}$$

and u is the eccentric anomaly as a function of time via (9.20).

9.2. Variational equations of the circular Sitnikov problem. The variational equations (9.9) of the circular Sitnikov problem (9.3) with $r = 1/2$ along a given periodic solution curve $(z(t), Z(t))$ were solved in [8]; therefore we will refer to the corresponding results in this paper when it is necessary.

9.3. Variational equations of the elliptic Sitnikov problem for small values of the eccentricity. We consider the elliptic Sitnikov problem (9.3) where $r(t)(1 - e\cos u)/2$ is the elliptic solution of the Kepler problem (9.2), $0 < e < 1$, and u is the eccentric anomaly, which is a function of t via equation (9.12).

If the eccentricity e is small, then $r(t)$ may be expanded in terms of the mean anomaly M and of the eccentricity e, and $r(t) = (1 - e\cos M)/2 + O(e^2)$ (see, for instance, [3]). Thus, system (9.3) may be written as

$$\dot{z} = Z, \qquad \dot{Z} = -\frac{z}{(z^2 + 1/4)^{3/2}} - e\left[\frac{3}{4}\frac{z}{(z^2 + 1/4)^{5/2}}\cos M\right] + O(e^2). \tag{9.22}$$

Let $(z(t), Z(t))$ be a periodic solution of system (9.22). If the eccentricity e is sufficiently small, then by the Poincaré expansion theorem (see, for instance, [20] or [13]) $(z(t), Z(t))$ may be expanded in power series of e and

$$(z(t), Z(t)) = (z_{(0)}(t) + z_{(1)}(t)e + O(e^2), Z_{(0)}(t) + Z_{(1)}(t)e + O(e^2)),$$

where $(z_{(0)}(t), Z_{(0)}(t))$ is a given solution of the circular Sitnikov problem (or, equivalently, a solution of (9.22) for $e = 0$).

We analyze here the solution of the variational equations of the elliptic Sitnikov problem (9.3) along the solution curve $(z(t), Z(t))$ for $e > 0$ sufficiently small. These variational equations are given by the matrix differential equation

$$\frac{d}{dt}A_4 = \begin{pmatrix} 0 & 1 \\ \bar{b}(t) & 0 \end{pmatrix} A_4,$$

with initial condition $A_4(0) = I$ (the 2×2 identity matrix), where

$$\begin{aligned}\bar{b}(t) &= \frac{2z^2(t) - 1/4}{(z^2(t) + 1/4)^{5/2}} + e\left[\frac{3}{4}\frac{4z^2(t) - 1/4}{(z^2(t) + 1/4)^{7/2}}\cos M\right] + O(e^2)\\ &= \frac{2{z_{(0)}}^2(t) - 1/4}{({z_{(0)}}^2(t) + 1/4)^{5/2}} + eF(t) + O(e^2),\end{aligned}$$

and

$$F(t) = \frac{-6{z_{(0)}}^3(t)z_{(1)}(t) + 9z_{(0)}(t)z_{(1)}(t)/4 + 3\cos M(4{z_{(0)}}^2(t) - 1/4)/4}{({z_{(0)}}^2(t) + 1/4)^{7/2}}.$$

Thus the derivatives $(\partial z/\partial z_0, \partial Z/\partial z_0)$ and $(\partial z/\partial Z_0, \partial Z/\partial Z_0)$ are given by the solution of system

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = \left(\frac{2{z_{(0)}}^2(t) - 1/4}{({z_{(0)}}^2(t) + 1/4)^{5/2}} + F(t)e + O(e^2)\right)x,$$

with initial conditions $x(0) = 1$, $y(0) = 0$ and $x(0) = 0$, $y(0) = 1$, respectively. By the Poincaré expansion theorem this solution may be expanded in power series of e and

$$\begin{pmatrix} \dfrac{\partial z}{\partial z_0}(t) & \dfrac{\partial z}{\partial Z_0}(t) \\ \dfrac{\partial Z}{\partial z_0}(t) & \dfrac{\partial Z}{\partial Z_0}(t) \end{pmatrix} = \begin{pmatrix} \displaystyle\sum_{n=0}^{\infty} x_{1(n)}(t)e^n & \displaystyle\sum_{n=0}^{\infty} x_{2(n)}(t)e^n \\ \displaystyle\sum_{n=0}^{\infty} y_{1(n)}(t)e^n & \displaystyle\sum_{n=0}^{\infty} y_{2(n)}(t)e^n \end{pmatrix}, \tag{9.23}$$

where $\left(\begin{smallmatrix} x_{1(0)}(t) & x_{2(0)}(t) \\ y_{1(0)}(t) & y_{2(0)}(t) \end{smallmatrix}\right)$ is the solution of the variational equations (9.9) of the circular Sitnikov problem along the solution curve $(z_{(0)}(t), Z_{(0)}(t))$ and

$$\begin{pmatrix} x_{1(n)}(t) & x_{2(n)}(t) \\ y_{1(n)}(t) & y_{2(n)}(t) \end{pmatrix} = \begin{pmatrix} \dfrac{\partial^n}{\partial e^n}\left(\dfrac{\partial z}{\partial z_0}\right)(t)\Bigg|_{e=0} & \dfrac{\partial^n}{\partial e^n}\left(\dfrac{\partial z}{\partial Z_0}\right)(t)\Bigg|_{e=0} \\ \dfrac{\partial^n}{\partial e^n}\left(\dfrac{\partial Z}{\partial z_0}\right)(t)\Bigg|_{e=0} & \dfrac{\partial^n}{\partial e^n}\left(\dfrac{\partial Z}{\partial Z_0}\right)(t)\Bigg|_{e=0} \end{pmatrix}.$$

We remark that the solution of the variational equations (9.9) of the circular Sitnikov problem along the solution curve $(z_{(0)}(t), Z_{(0)}(t))$ is unbounded when t goes to infinity. Therefore, with a fixed value of e, (9.23) is valid only for t less than a constant which depends on the value of e.

10. Continuation of periodic orbits from the reduced circular Sitnikov problem to the reduced isosceles problem. In this section we will use the analytic continuation method of Poincaré to continue the periodic orbits of the reduced circular Sitnikov problem to symmetric periodic orbits of the reduced isosceles problem for $\mu > 0$ sufficiently small.

Choosing conveniently the origin of time, the periodic orbit of the reduced circular Sitnikov problem with period $T > \pi/\sqrt{2}$ is the orbit associated to the periodic solution with initial conditions $\varphi_{1/4}(t; r_0 = 1/2, \dot r_0 = 0, z_0 = 0, \dot z_0 = \dot z_0^* = \sqrt{2h+4}, \mu = 0)$. Here $h \in (-2, 0)$ is the energy of the periodic orbit of period $T \in (\pi/\sqrt{2}, \infty)$ (see Theorem 7.5 for details). We remark that the notation used here is the one defined in section 8.

Since the reduced isosceles problem is autonomous, if we continue using different initial conditions defining the same periodic orbit, then we will obtain the same continued periodic orbits. So, it will be sufficient to continue periodic solutions with initial conditions $\varphi_{1/4}(t; 1/2, 0, 0, \dot z_0^*, 0)$ for $-2 < h < 0$. We note that these periodic solutions are doubly symmetric, so we can investigate their continuation to periodic solutions of the reduced isosceles problem for $\mu > 0$ small that are either doubly symmetric, r-symmetric, or t-symmetric. Here we analyze only the continuation to doubly symmetric periodic solutions. We have also analyzed the continuation to r- and to t-symmetric periodic solutions, but these two types of continuation provide again the same families of doubly symmetric periodic orbits of the reduced isosceles problem for $\mu > 0$ small (for details see [6]).

By Proposition 5.3(1), if we can find initial conditions r_0 and $\dot z_0$ such that the solution $\varphi_{1/4}(t; r_0, 0, 0, \dot z_0, \mu) = (r(t; r_0, \dot z_0, \mu), \dot r(t; r_0, \dot z_0, \mu), z(t; r_0, \dot z_0, \mu), \dot z(t; r_0, \dot z_0, \mu))$ of the reduced isosceles problem (3.1) with $c = 1/4$ satisfies

$$\dot r(\tau/4; r_0, \dot z_0, \mu) = 0, \qquad \dot z(\tau/4; r_0, \dot z_0, \mu) = 0, \tag{10.1}$$

and $\dot r$,$\dot z$ are not simultaneously zero for $t \in (0, \tau/4)$, then $\varphi_{1/4}(t; r_0, 0, 0, \dot z_0, \mu)$ is a doubly symmetric periodic solution with period τ.

Observe that $\tau = T = T(h)$, $r_0 = 1/2$, $\dot z_0 = \dot z_0^* = \sqrt{2h+4}$, and $\mu = 0$ is a solution of (10.1) for each $-2 < h < 0$. It corresponds to the doubly symmetric periodic solution $\varphi_{1/4}(t; 1/2, 0, 0, \dot z_0^*, 0)$ of the reduced circular Sitnikov problem. Our aim is to find solutions of (10.1) near the known solution $\tau = T$, $r_0 = 1/2$, $\dot z_0 = \dot z_0^*$, and $\mu = 0$. For this goal, we will apply the implicit function theorem to (10.1) in a neighborhood of that point, choosing $(r_0, \dot z_0)$ as the dependent variables and (μ, τ) as the independent ones.

We note that there are five other choices for the dependent (independent) variables. Since we want to continue periodic solutions from $\mu = 0$ to $\mu > 0$ small, we are interested in solutions of (10.1) depending on μ. So, the other possible choices for the independent variables are (μ, r_0) and $(\mu, \dot{z}_0)$. Since the reduced isosceles problem possesses the first integral of the energy, we also could be interested in expressing the solutions of (10.1) as a function of μ and of the energy $\widetilde{h}$. We have analyzed these other possible choices for the independent variables, then saw that the implicit function theorem using either (μ, r_0) or $(\mu, \widetilde{h})$ as the independent variables cannot be applied to this problem because the corresponding determinant vanishes. Moreover, if we apply the implicit function theorem using $(\mu, \dot{z}_0)$ as the independent variables, we obtain the same solutions of (10.1) as we do using (μ, τ). The difference is that these solutions are parameterized by $(\mu, \dot{z}_0)$ instead of (μ, τ).

We apply the implicit function theorem to system (10.1) in a neighborhood of the point $\tau = T$, $r_0 = 1/2$, $\dot{z}_0 = \dot{z}_0^*$, and $\mu = 0$, choosing μ and τ as the independent variables. If

$$\det \begin{pmatrix} \dfrac{\partial \dot{r}(\tau/4; r_0, \dot{z}_0, \mu)}{\partial r_0} & \dfrac{\partial \dot{r}(\tau/4; r_0, \dot{z}_0, \mu)}{\partial \dot{z}_0} \\ \dfrac{\partial \dot{z}(\tau/4; r_0, \dot{z}_0, \mu)}{\partial r_0} & \dfrac{\partial \dot{z}(\tau/4; r_0, \dot{z}_0, \mu)}{\partial \dot{z}_0} \end{pmatrix}_{\big|(\mu=0, \tau=T, r_0=1/2, \dot{z}_0=\dot{z}_0^*)} \neq 0, \tag{10.2}$$

then for each (μ, τ) in a sufficiently small neighborhood of $(0, T)$, there exist two unique functions $r_0 = r_0(\mu, \tau)$ and $\dot{z}_0 = \dot{z}_0(\mu, \tau)$ such that $r_0(0, T) = 1/2$, $\dot{z}_0(0, T) = \dot{z}_0^*$, and r_0, $\dot{z}_0$ satisfy system (10.1). We note that the negative values of μ do not have physical meaning. Therefore, if condition (10.2) is satisfied, then for each (μ, τ) in a sufficiently small neighborhood of $(0, T)$ with $\mu \geqslant 0$, $\varphi_{1/4}(t; r_0(\mu, \tau), 0, 0, \dot{z}_0(\mu, \tau), \mu)$ is a doubly symmetric periodic solution of the reduced isosceles problem (3.1) for $c = 1/4$ with period τ. Since the functions that appear in system (10.1) are analytic, the functions $r_0(\mu, \tau)$ and $\dot{z}_0(\mu, \tau)$ are also analytic and may be expanded in power series of μ and $\overline{\tau} = \tau - T$ in U, a sufficiently small neighborhood of $(0, 0)$; that is, $r_0 = 1/2 + O(\mu, \overline{\tau})$ and $\dot{z}_0 = \dot{z}_0^* + O(\mu, \overline{\tau})$.

Now we compute the value of the determinant (10.2). The derivatives that appear in this determinant are obtained by evaluating at time $t = T/4$ the corresponding solution of the variational equations of the reduced restricted isosceles problem (6.2) for $c = 1/4$ along the solution curve $\varphi_{1/4}(t; 1/2, 0, 0, \dot{z}_0^*, 0)$. These variational equations were solved in section 9. Then, from (9.8),

$$\left. \frac{\partial \dot{r}(\tau/4; r_0, \dot{z}_0, \mu)}{\partial \dot{z}_0} \right|_{(\mu=0, \tau=T, r_0=1/2, \dot{z}_0=\dot{z}_0^*)} = 0.$$

The value of the derivative $\partial \dot{r}(\tau/4; r_0, \dot{z}_0, \mu)/\partial r_0$ evaluated at $\mu = 0$, $\tau = T$, $r_0 = 1/2$, and $\dot{z}_0 = \dot{z}_0^*$ can be obtained by evaluating at $t = T/4$ the corresponding solution of the variational equations of the Kepler problem (9.2), with $e = 0$, along the solution curve $(r(t) = 1/2, \dot{r}(t) = 0)$. Thus, by Proposition 9.2, we get

$$\left. \frac{\partial \dot{r}(\tau/4; r_0, \dot{z}_0, \mu)}{\partial r_0} \right|_{(\mu=0 \tau=T, r_0=1/2, \dot{z}_0=\dot{z}_0^*)} = -\sin(T/4),$$

which is different from zero if and only if the period T is a nonmultiple of 4π.

It remains only to find the value of $\partial \dot{z}(\tau/4; r_0, \dot{z}_0, \mu)/\partial \dot{z}_0$ at $\mu = 0$, $\tau = T$, $r_0 = 1/2$, and $\dot{z}_0 = \dot{z}_0^*$. This value can be obtained by evaluating at $t = T/4$ the

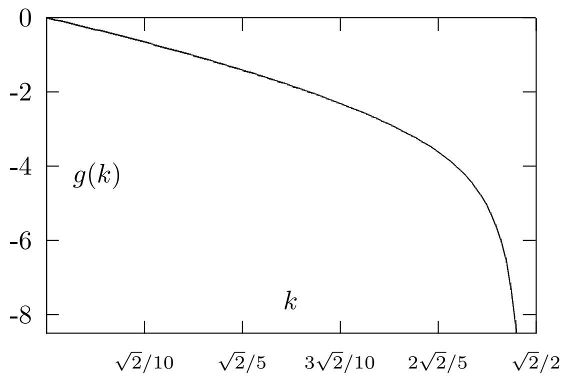

FIG. 10.1. *The graphic of* $g(k)$.

corresponding solution of the variational equations of the circular Sitnikov problem along the solution curve $(z(t;1/2,0,0,\dot z_0^*,0), \dot z(t;1/2,0,0,\dot z_0^*,0))$. The solution of those variational equations is given by formula (B.12) of [8]. In particular, the derivative $\partial\dot z(t;r_0,\dot z_0,\mu)/\partial\dot z_0$ evaluated at $\mu=0$, $r_0=1/2$, and $\dot z_0=\dot z_0^*$ is

$$\begin{aligned}
&\frac{(1-2k^2\,\mathrm{sn}^2\nu)^2}{(2k^2-1)^2k'^2}\Bigg[-k^2\,\mathrm{sn}^2\nu\,\mathrm{cn}\,\nu+\mathrm{dn}^2\nu\,\mathrm{cn}\,\nu-\mathrm{sn}\,\nu\,\mathrm{dn}\,\nu\Bigg(k'^2(k^2+1)\nu\\
&\quad-(2k^2k'^2+1)E(\nu)-3k^2k'^2\Pi(\nu,2k^2)+4k^4k'^2\frac{\mathrm{sn}\,\nu\,\mathrm{cn}\,\nu\,\mathrm{dn}\,\nu}{1-2k^2\,\mathrm{sn}^2\nu}\Bigg)\\
&\quad+\mathrm{cn}\,\nu\Bigg(k'^2(k^2+1)-(2k^2k'^2+1)\,\mathrm{dn}^2\nu-\frac{3k^2k'^2}{1-2k^2\,\mathrm{sn}^2\nu}\\
&\quad+4k^4k'^2\frac{(\mathrm{cn}^2\nu\,\mathrm{dn}^2\nu-\mathrm{sn}^2\nu\,\mathrm{dn}^2\nu-k^2\,\mathrm{sn}^2\nu\,\mathrm{cn}^2\nu)}{1-2k^2\,\mathrm{sn}^2\nu}\\
&\quad+16k^6k'^2\frac{\mathrm{sn}^2\nu\,\mathrm{cn}^2\nu\,\mathrm{dn}^2\nu}{(1-2k^2\,\mathrm{sn}^2\nu)^2}\Bigg)\Bigg],
\end{aligned}\tag{10.3}$$

where ν is a function of t via Lemma 7.2(1), $k=\sqrt{2+h}/2$, and $k'=\sqrt{1-k^2}$.

By Lemmas 7.2(1) and 7.4, we have that $\nu(0)=0$ and $\nu(T/4)=K$, respectively. Then, by formula 122.02 of [4] we have that $\mathrm{sn}\,\nu(T/4)=1$, $\mathrm{cn}\,\nu(T/4)=0$, $\mathrm{dn}\,\nu(T/4)=k'$, and by formula (A.5) of [8] we have that $E(\nu(T/4))=E$ and $\Pi(\nu(T/4),2k^2)=\Pi(2k^2,k)$. Therefore,

$$\frac{\partial\dot z(\tau/4;r_0,\dot z_0,\mu)}{\partial\dot z_0}\Bigg|_{(\mu=0,\tau=T,r_0=1/2,\dot z_0=\dot z_0^*)}=-\frac{1}{k'}g(k),$$

where

$$g(k)=k'^2(k^2+1)K-(2k^2k'^2+1)E-3k^2k'^2\Pi(2k^2,k).\tag{10.4}$$

Since $-2<h<0$, we have that $k\in(0,\sqrt{2}/2)$. We plot the function $g(k)$ in the range $0<k<\sqrt{2}/2$, obtaining Figure 10.1. Therefore $g(k)$ is always different from zero except when $k=0$, but this case is not considered here because it corresponds to the equilibrium point of the reduced circular Sitnikov problem.

In short, if the period $T=T(h)$ is a nonmultiple of 4π, then determinant (10.2) is different from zero. This proves the following theorem.

THEOREM 10.1. *For any $T > \pi/\sqrt{2}$, with $T \neq 4\pi n$ for all $n \in \mathbb{N}$, the periodic orbit of the reduced circular Sitnikov problem with period T can be continued to a 2-parameter family (on μ and τ) of doubly symmetric periodic orbits of the reduced isosceles problem* (3.1) *with angular momentum $c = 1/4$, which have period τ for (μ, τ) in a sufficiently small neighborhood of $(0, T)$ with $\mu \geqslant 0$.*

10.1. Remarks. We note that Theorem 10.1 also gives periodic orbits of the reduced isosceles problem for $\mu = 0$. One might think that this theorem could be used to find new symmetric periodic orbits of the reduced elliptic Sitnikov problem. But this is not the case because the symmetric periodic orbits for $\mu = 0$ that we obtain in this way are periodic orbits of the reduced circular Sitnikov problem, which are already known. This follows from the fact that the functions $r_0(\mu, \tau)$ and $\dot{z}_0(\mu, \tau)$ are unique and that $\varphi_{1/4}(t; r_0 = 1/2, 0, 0, \dot{z}_0 = \sqrt{2h(\tau) + 4}, 0)$ is a periodic solution of the reduced circular Sitnikov problem.

On the other hand, Theorem 10.1 does not allow us to continue the periodic orbits of the reduced circular Sitnikov problem that have period T that is a multiple of 4π. Later on, in section 12, we will see that these periodic solutions can be continued in two steps to two different families of doubly symmetric periodic solutions of the reduced isosceles problem (3.1) with angular momentum $c = 1/4$ and $\mu > 0$ sufficiently small, having period τ near T (see Theorem 12.8). The fact that the continuation is to two families explains why we have not been able here to continue these periodic orbits using only the implicit function theorem.

Often when we analyze a problem of continuation of periodic solutions we are interested in families of periodic solutions with the same period or with the same energy (these last families are called *isoenergetic families*). We could also consider families of periodic solutions with a fixed initial condition. In order to obtain these kinds of families in our problem we would fix one of the variables (it could be T, $\widetilde{h}$, r_0, or $\dot{z}_0$) in system (10.1), and then we would continue, in function of μ, the known periodic solutions of the reduced circular Sitnikov problem. We have done that and seen that the periodic solutions of the reduced circular Sitnikov problem with period T, nonmultiple of 4π, can be continued to a 1-parameter family (on μ) of doubly symmetric periodic solutions of the reduced isosceles problem having fixed period T, and another 1-parameter family having fixed initial condition $\dot{z}_0 = \dot{z}_0^*$. Clearly these two families are contained in the 2-parameter family of doubly symmetric periodic orbits of the reduced isosceles problem obtained in Theorem 10.1. Finally, the continuation fixing either the initial condition r_0 or the energy $\widetilde{h}$ is not possible because the corresponding determinants vanish.

Theorem 10.1 is improved by the following result.

THEOREM 10.2. *For any interval $[T_1, T_2]$ with $T_1 > \pi/\sqrt{2}$ and such that $4\pi n \notin [T_1, T_2]$ for all $n \in \mathbb{N}$, there exist $\mu_0 > 0$ and two unique analytic functions $r_0(\mu, \tau)$ and $\dot{z}_0(\mu, \tau)$, defined for all $\mu \in [0, \mu_0)$ and $\tau \in [T_1, T_2]$, such that $\varphi_{1/4}(t; r_0(\mu, \tau), 0, 0, \dot{z}_0(\mu, \tau), \mu)$ is a double symmetric periodic solution, with period τ, of the reduced isosceles problem* (3.1) *with angular momentum $c = 1/4$. Moreover $r_0(0, \tau) = 1/2$ and $\dot{z}_0(0, \tau) = \sqrt{2h(\tau) + 4}$, where $h(\tau)$ is the value of the energy of the periodic orbit of the circular Sitnikov problem having period τ.*

Proof. Fixed $\tau^* \in [T_1, T_2]$, the implicit function theorem assures the existence of two unique analytic functions $r_0(\mu, \tau)$ and $\dot{z}_0(\mu, \tau)$ for (μ, τ) in a sufficiently small neighborhood of $(0, \tau^*)$. Due to the compactness of $[T_1, T_2]$ and the uniqueness of $r_0(\mu, \tau)$ and $\dot{z}_0(\mu, \tau)$, we can find $\mu_0 > 0$ such that, for $0 \leqslant \mu < \mu_0$, these functions are defined for all $\tau^* \in [T_1, T_2]$, which proves the result. ∎

11. Continuation of symmetric periodic orbits from the reduced elliptic Sitnikov problem to the reduced isosceles problem. In this section we will continue the known symmetric periodic solutions of the reduced elliptic Sitnikov problem with eccentricity e (meaning the symmetric periodic solutions given in section 8) to symmetric periodic solutions of the reduced isosceles problem with $c = c_e$ and $\mu > 0$ sufficiently small. In particular we will prove the following result.

THEOREM 11.1. *Let γ_{e_0} be a symmetric periodic orbit of the reduced elliptic Sitnikov problem with eccentricity e_0 given by Theorems 8.2 or 8.3 that has period $\tau^\diamond = 2\pi p = qT$ for fixed values of $p, q \in \mathbb{N}$ coprime with $p > q/(2\sqrt{2})$. If the eccentricity e_0 is sufficiently small, then γ_{e_0} can be continued to a 2-parameter family (on μ and τ) of symmetric periodic orbits of the reduced isosceles problem (3.1) with angular momentum $c = \sqrt{1-{e_0}^2}/4$ and $\mu \geqslant 0$ that have period τ for (μ, τ) in a sufficiently small neighborhood of $(0, \tau^\diamond)$. Moreover the continued periodic orbits satisfy the same symmetry as the initial orbit γ_{e_0}.*

Apart from the symmetric periodic orbits of the reduced elliptic Sitnikov problem given by Theorems 8.2 and 8.3, we know the existence of infinitely many symmetric periodic orbits of the reduced elliptic Sitnikov problem for all $0 < e < 1$ (see Propositions 12 and 15 in [7]); unfortunately we do not know analytical expressions for their initial conditions. Nevertheless we will give sufficient conditions in order to continue an arbitrary symmetric periodic orbit of the reduced elliptic Sitnikov problem to symmetric periodic orbits of the reduced isosceles problem for $\mu > 0$ sufficiently small.

We start analyzing the continuation of doubly symmetric periodic orbits of the reduced elliptic Sitnikov problem, after which we will analyze the continuation of r- and t-symmetric periodic orbits.

Choosing conveniently the origin of time, the symmetric periodic orbits of the reduced elliptic Sitnikov problem can be seen as the orbits associated to symmetric periodic solutions with initial conditions either $\varphi_{c_e}(t; r_0 = r_0^\diamond = (1 \pm e)/2, \dot r_0 = 0, z_0 = 0, \dot z_0 = \dot z_0^\diamond, \mu = 0)$ or $\varphi_{c_e}(t; r_0 = r_0^\diamond = (1 \pm e)/2, \dot r_0 = 0, z_0 = z_0^\diamond, \dot z_0 = 0, \mu = 0)$. So, we will study only the continuation of symmetric periodic solutions of these types. Of course, if we continue different initial conditions defining the same periodic orbit, then we will obtain the same periodic orbit of the reduced isosceles problem.

11.1. Continuation of doubly symmetric periodic solutions. As in section 10, by Proposition 5.3(1), the solution $\varphi_{c_e}(t; r_0, 0, 0, \dot z_0, \mu) = (r(t; r_0, \dot z_0, \mu), \dot r(t; r_0, \dot z_0, \mu), z(t; r_0, \dot z_0, \mu), \dot z(t; r_0, \dot z_0, \mu))$ is a doubly symmetric periodic solution of the reduced isosceles problem (3.1) with $c = c_e$ having period τ if it satisfies

$$\dot r(\tau/4; r_0, \dot z_0, \mu) = 0, \qquad \dot z(\tau/4; r_0, \dot z_0, \mu) = 0, \tag{11.1}$$

and $\dot r$, $\dot z$ are not simultaneously zero for $t \in (0, \tau/4)$.

Let $\varphi_{c_e}(t; r_0 = r_0^\diamond = (1 \pm e)/2, \dot r_0 = 0, z_0 = 0, \dot z_0 = \dot z_0^\diamond, \mu = 0)$ be a doubly symmetric periodic solution of the reduced elliptic Sitnikov problem for a fixed $0 < e < 1$ and let $\tau^\diamond = 2\pi p$, with $p \in \mathbb{N}$ even, be its period. This is equivalent to saying that $\tau = \tau^\diamond$, $r_0 = r_0^\diamond$, $\dot z_0 = \dot z_0^\diamond$, and $\mu = 0$ is a solution of system (11.1).

Applying the implicit function theorem to (11.1) in a neighborhood of the point $\tau = \tau^\diamond$, $r_0 = r_0^\diamond$, $\dot z_0 = \dot z_0^\diamond$, and $\mu = 0$, and choosing μ and τ as the independent variables, if

$$\det \begin{pmatrix} \dfrac{\partial \dot r(\tau/4; r_0, \dot z_0, \mu)}{\partial r_0} & \dfrac{\partial \dot r(\tau/4; r_0, \dot z_0, \mu)}{\partial \dot z_0} \\ \dfrac{\partial \dot z(\tau/4; r_0, \dot z_0, \mu)}{\partial r_0} & \dfrac{\partial \dot z(\tau/4; r_0, \dot z_0, \mu)}{\partial \dot z_0} \end{pmatrix}_{\big|(\mu=0, \tau=\tau^\diamond, r_0=r_0^\diamond, \dot z_0=\dot z_0^\diamond)} \neq 0,$$

then for each $(\mu, \overline{\tau} = \tau - \tau^\diamond)$ in a sufficiently small neighborhood W_d of $(0,0)$ with $\mu \geqslant 0$, we can find two unique analytic functions $r_0(\mu,\tau) = r_0^\diamond + O(\mu,\overline{\tau})$ and $\dot z_0(\mu,\tau) = \dot z_0^\diamond + O(\mu,\overline{\tau})$ such that $\varphi_{c_e}(t; r_0^\diamond + O(\mu,\overline{\tau}), 0, 0, \dot z_0^\diamond + O(\mu,\overline{\tau}), \mu)$ is a doubly symmetric periodic solution of period τ for the reduced isosceles problem (3.1) for $c = c_e$ and $\mu \geqslant 0$ small enough.

The derivatives that appear in this determinant are obtained by evaluating at time $t = \tau^\diamond/4$ the corresponding solution of the variational equations of the reduced restricted isosceles problem (6.2) for $c = c_e$ along the solution curve $\varphi_{c_e}(t; r_0^\diamond, 0, 0, \dot z_0^\diamond, 0)$ with $r_0^\diamond = (1 \pm e)/2$. The solution of these variational equations has been studied in section 9.1.

By (9.8), the derivative $\partial \dot r(\tau/4; r_0, \dot z_0, \mu)/\partial \dot z_0$ evaluated at $\mu = 0$, $\tau = \tau^\diamond$, $r_0 = r_0^\diamond$, and $\dot z_0 = \dot z_0^\diamond$ equals zero. The value of the derivative $\partial \dot r(\tau/4; r_0, \dot z_0, \mu)/\partial r_0$ evaluated at $\mu = 0$, $\tau = \tau^\diamond$, $r_0 = r_0^\diamond$, and $\dot z_0 = \dot z_0^\diamond$ can be obtained by evaluating at $t = T/4$ the corresponding solution of the variational equations of the Kepler problem (9.2) along the solution curve $(r(t; r_0^\diamond, \dot z_0^\diamond, 0), \dot r(t; r_0^\diamond, \dot z_0^\diamond, 0))$.

If $r_0^\diamond = (1-e)/2$—that is, $t = 0$ corresponds to the minimum value of $r(t; r_0^\diamond, \dot z_0^\diamond, 0)$—then from Kepler's equation (9.18), $u(\tau^\diamond/4) = u(m\pi) = m\pi$. Moreover, since p is even, $p = 2m$ for some $m \in \mathbb{N}$. Therefore, from (9.19),

$$\left.\frac{\partial \dot r(\tau/4; r_0, \dot z_0, \mu)}{\partial r_0}\right|_{(\mu=0,\tau=\tau^\diamond, r_0=\frac{1-e}{2}, \dot z_0 = \dot z_0^\diamond)} = \frac{3e^2 m\pi((-1)^m - e)}{(1-(-1)^m e)^3 (1-e)^2}, \tag{11.2}$$

which is different from zero because $e \neq 0$ and $e \neq 1$.

If $r_0^\diamond = (1+e)/2$—that is, $t = 0$ corresponds to the maximum value of $r(t; r_0^\diamond, \dot z_0^\diamond, 0)$—then from Kepler's equation (9.20), $u(\tau^\diamond/4) = u(m\pi) = (m+1)\pi$. Thus by (9.21),

$$\left.\frac{\partial \dot r(\tau/4; r_0, \dot z_0, \mu)}{\partial r_0}\right|_{(\mu=0,\tau=\tau^\diamond, r_0=\frac{1+e}{2}, \dot z_0 = \dot z_0^\diamond)} = \frac{3e^2 m\pi(e - (-1)^{m+1})}{(1-(-1)^{m+1} e)^3 (1+e)^2}, \tag{11.3}$$

which is also different from zero. In short, we have proved the following result.

THEOREM 11.2. *Let $\varphi_{c_e}(t; r_0 = r_0^\diamond = (1 \pm e)/2, \dot r_0 = 0, z_0 = 0, \dot z_0 = \dot z_0^\diamond, \mu = 0)$ be a doubly symmetric periodic solution of the reduced elliptic Sitnikov problem for a fixed $0 < e < 1$ and let $\tau^\diamond = 2\pi p$ with $p \in \mathbb{N}$ even be its period. If*

$$\left.\frac{\partial \dot z(\tau/4; r_0, \dot z_0, \mu)}{\partial \dot z_0}\right|_{(\mu=0,\tau=\tau^\diamond, r_0=r_0^\diamond, \dot z_0 = \dot z_0)} \neq 0, \tag{11.4}$$

then this solution can be analytically continued to a 2-parameter family (on μ and τ) $\varphi_{c_e}(t; r_0 = r_0^\diamond + O(\mu,\overline{\tau}), \dot r_0 = 0, z_0 = 0, \dot z_0 = \dot z_0^\diamond + O(\mu,\overline{\tau}), \mu)$ of doubly symmetric periodic solutions of the reduced isosceles problem, with angular momentum $c = c_e$ and $\mu \geqslant 0$, that have period τ for $(\mu, \overline{\tau} = \tau - \tau^\diamond) \in W_d$, with W_d a sufficiently small neighborhood of $(0,0)$.

11.1.1. Application of Theorem 11.2. Now we apply Theorem 11.2 to continue the doubly symmetric periodic solutions of the reduced elliptic Sitnikov problem given by Theorem 8.2. Let $\varphi_{c_e}(t; r_0 = r_0^\diamond = (1 \pm e)/2, \dot r_0 = 0, z_0 = 0, \dot z_0 = \dot z_0^\diamond = \dot z_0^* + O(e), \mu = 0)$, with $\dot z_0^* = \pm\sqrt{2h+4}$, be one of these periodic solutions for a fixed $e > 0$ sufficiently small and fixed $p, q \in \mathbb{N}$ coprime with p even and $p > q/(2\sqrt{2})$. By Theorem 11.2, this doubly symmetric periodic solution can be continued to doubly symmetric periodic solutions of the reduced isosceles problem for $\mu > 0$ if (11.4) holds. The value of the derivative (11.4) is obtained from the solution, evaluated at $t = \tau^\diamond/4$, of the variational equations of the elliptic Sitnikov problem (9.3) along the solution curve $(z(t), \dot{z}(t)) = (z(t; r_0^\diamond, \dot{z}_0 = \dot{z}_0^\diamond = \dot{z}_0^* + O(e), 0), \dot{z}(t; r_0^\diamond, \dot{z}_0 = \dot{z}_0^\diamond = \dot{z}_0^* + O(e), 0))$. We note that if the eccentricity e is sufficiently small, then by the Poincaré expansion theorem, the solution $(z(t), \dot{z}(t))$ may be expanded in power series of $\dot{z}_0^\diamond - \dot{z}_0^*$ and e. Since $\dot{z}_0^\diamond - \dot{z}_0^* = O(e)$, we have that $(z(t), \dot{z}(t)) = (z_{(0)}(t) + O(e), \dot{z}_{(0)}(t) + O(e))$, where $(z_{(0)}(t), \dot{z}_{(0)}(t))$ is the solution of the circular Sitnikov problem with initial conditions $z_{(0)}(0) = 0$ and $\dot{z}_{(0)}(0) = \dot{z}_0^*$. So, the solution of the variational equations of the elliptic Sitnikov problem along that solution curve $(z(t), \dot{z}(t))$ is given by the solution of the variational equations of the reduced circular Sitnikov problem along the solution curve $(z_{(0)}(t), \dot{z}_{(0)}(t))$ plus terms of at least order one in e (see section 9.3). Since $\dot{z}_0^* = \pm\sqrt{2h+4}$, the solution of these last variational equations is given by formula (B.12) of [8].

We assume that e is small enough so that (9.23) is valid at least for $0 \leqslant t \leqslant \tau^\diamond/4$. From formula (B.12) of [8] and (9.23), the derivative $\partial\dot{z}(t; r_0, \dot{z}_0, \mu)/\partial\dot{z}_0$ evaluated at $\mu = 0$, $r_0 = r_0^\diamond$, and $\dot{z}_0 = \dot{z}_0^\diamond$ is given by (10.3) plus terms of at least order one in e. On the other hand, from Lemmas 7.2(1) and 7.4, we have that $\nu(0) = 0$ and $\nu(\tau/4) = qK$, respectively. We consider that $q = 2l + 1$ for some $l \in \mathbb{N}$ (we note that q is odd because p is even and p and q are coprime). By formulas 122.02 and 122.04 of [4] we have that $\operatorname{sn}\nu(\tau/4) = (-1)^l$, $\operatorname{cn}\nu(\tau/4) = 0$, $\operatorname{dn}\nu(\tau/4) = k'$; moreover by formula (A.5) of [8] we have that $E(\nu(\tau/4)) = qE$ and $\Pi(\nu(\tau/4), 2k^2) = q\Pi(2k^2, k)$. Hence

$$\left.\frac{\partial\dot{z}(\tau/4; r_0, \dot{z}_0, \mu)}{\partial\dot{z}_0}\right|_{(\mu=0,\tau=\tau^\diamond,r_0=r_0^\diamond,\dot{z}_0=\dot{z}_0^\diamond)} = -\frac{(-1)^l q}{k'} g(k) + O(e),$$

where $l \in \mathbb{N}$ is such that $q = 2l + 1$, and $g(k)$ is given by (10.4). Since $g(k)$ is always different from zero, if the eccentricity e is small enough, then the derivative $\partial\dot{z}(\tau/4; r_0, \dot{z}_0, \mu)/\partial\dot{z}_0$ evaluated at $\mu = 0$, $\tau = \tau^\diamond$, $r_0 = r_0^\diamond$, and $\dot{z}_0 = \dot{z}_0^\diamond$ is different from zero. Thus we have the following result.

COROLLARY 11.3. *For fixed $e > 0$ sufficiently small, let $\varphi_{c_e}(t; r_0 = r_0^\diamond = (1 \pm e)/2, \dot{r}_0 = 0, z_0 = 0, \dot{z}_0 = \dot{z}_0^\diamond = \pm\sqrt{2h+4} + O(e), \mu = 0)$ be one of the doubly symmetric periodic solutions of the reduced elliptic Sitnikov problem given by Theorem* 8.2 *that has period $\tau^\diamond = 2\pi p = qT$ for given values of $p, q \in \mathbb{N}$ coprime with p even and $p > q/(2\sqrt{2})$. Then this solution can be analytically continued to a* 2*-parameter family (on μ and τ) $\varphi_{c_e}(t; r_0 = r_0^\diamond + O(\mu, \overline{\tau}), \dot{r}_0 = 0, z_0 = 0, \dot{z}_0 = \dot{z}_0^\diamond + O(\mu, \overline{\tau}), \mu)$ of doubly symmetric periodic solutions of the reduced isosceles problem, with angular momentum $c = c_e$ and $\mu \geqslant 0$, that have period τ for $(\mu, \overline{\tau} = \tau - \tau^\diamond) \in W_d$, with W_d a sufficiently small neighborhood of* $(0, 0)$.

11.2. Continuation of *r*-symmetric periodic solutions. By Proposition 5.1, $\varphi_{c_e}(t; r_0, 0, 0, \dot{z}_0, \mu)$ is an r-symmetric periodic solution of the reduced isosceles problem (3.1) with $c = c_e$ having period τ if it satisfies

$$\dot{r}(\tau/2; r_0, \dot{z}_0, \mu) = 0\ , \qquad z(\tau/2; r_0, \dot{z}_0, \mu) = 0, \tag{11.5}$$

and $\dot{r}$, z are not simultaneously zero for $t \in (0, \tau/2)$.

Let $\varphi_{c_e}(t; r_0 = r_0^\diamond = (1 \pm e)/2, \dot{r}_0 = 0, z_0 = 0, \dot{z}_0 = \dot{z}_0^\diamond, \mu = 0)$ be an r-symmetric periodic solution of the reduced elliptic Sitnikov problem for a fixed $0 < e < 1$ and let $\tau^\diamond = 2\pi p$ with $p \in \mathbb{N}$ be its period. Or, equivalently, let $\tau = \tau^\diamond$, $r_0 = r_0^\diamond$, $\dot{z}_0 = \dot{z}_0^\diamond$, and $\mu = 0$ be a solution of system (11.5). Applying the implicit function theorem to system (11.5) in a neighborhood of that solution, choosing μ and τ as the independent variables, if

$$\det \begin{pmatrix} \dfrac{\partial \dot{r}(\tau/2; r_0, \dot{z}_0, \mu)}{\partial r_0} & \dfrac{\partial \dot{r}(\tau/2; r_0, \dot{z}_0, \mu)}{\partial \dot{z}_0} \\ \dfrac{\partial z(\tau/2; r_0, \dot{z}_0, \mu)}{\partial r_0} & \dfrac{\partial z(\tau/2; r_0, \dot{z}_0, \mu)}{\partial \dot{z}_0} \end{pmatrix}\Bigg|_{(\mu=0, \tau=\tau^\diamond, r_0=r_0^\diamond, \dot{z}_0=\dot{z}_0)} \neq 0,$$

then for each $(\mu, \overline{\tau} = \tau - \tau^\diamond)$ in a sufficiently small neighborhood W_r of $(0,0)$ with $\mu \geqslant 0$, we can find two unique analytic functions $r_0(\mu, \tau) = r_0^\diamond + O(\mu, \overline{\tau})$ and $\dot{z}_0(\mu, \tau) = \dot{z}_0^\diamond + O(\mu, \overline{\tau})$ such that $\varphi_{c_e}(t; r_0^\diamond + O(\mu, \overline{\tau}), 0, 0, \dot{z}_0^\diamond + O(\mu, \overline{\tau}), \mu)$ is an r-symmetric periodic solution of period τ for the reduced isosceles problem (3.1) for $c = c_e$ and $\mu \geqslant 0$ small.

The derivatives that appear in this determinant are obtained by evaluating at time $t = \tau^\diamond/2$ the corresponding solution of the variational equations of the reduced restricted isosceles problem (6.2) for $c = c_e$ along the solution curve $\varphi_{c_e}(t; r_0^\diamond, 0, 0, \dot{z}_0^\diamond, 0)$. Thus from (9.8),

$$\frac{\partial \dot{r}(\tau/2; r_0, \dot{z}_0, \mu)}{\partial \dot{z}_0}\Bigg|_{(\mu=0, \tau=\tau^\diamond, r_0=r_0^\diamond, \dot{z}_0=\dot{z}_0^\diamond)} = 0. \tag{11.6}$$

On the other hand, if $r_0^\diamond = (1-e)/2$, then from (9.18), $u(\tau^\diamond/2) = u(p\pi) = p\pi$; and if $r_0^\diamond = (1+e)/2$, then from (9.20), $u(\tau^\diamond/2) = u(p\pi) = (p+1)\pi$. Therefore, taking p instead of m in (11.2) and (11.3), we have that the derivative $\partial \dot{r}(\tau/2; r_0, \dot{z}_0, \mu)/\partial r_0$ evaluated at $\mu = 0$, $\tau = \tau^\diamond$, $\dot{z}_0 = \dot{z}_0^\diamond$, and $r_0 = (1-e)/2$ (respectively, $r_0 = (1+e)/2$) is given by

$$\frac{3e^2 p\pi((-1)^p - e)}{(1-(-1)^p e)^3 (1-e)^2} \qquad \left(\text{respectively, } \frac{3e^2 p\pi(e - (-1)^{p+1})}{(1-(-1)^{p+1} e)^3 (1+e)^2}\right), \tag{11.7}$$

which is different from zero. In short, we have proved the following result.

THEOREM 11.4. *Let $\varphi_{c_e}(t; r_0 = r_0^\diamond = (1 \pm e)/2, \dot{r}_0 = 0, z_0 = 0, \dot{z}_0 = \dot{z}_0^\diamond, \mu = 0)$ be an r-symmetric periodic solution of the reduced elliptic Sitnikov problem for a fixed $0 < e < 1$ and let $\tau^\diamond = 2\pi p$ with $p \in \mathbb{N}$ be its period. If*

$$\frac{\partial z(\tau/2; r_0, \dot{z}_0, \mu)}{\partial \dot{z}_0}\Bigg|_{(\mu=0, \tau=\tau^\diamond, r_0=r_0^\diamond, \dot{z}_0=\dot{z}_0^\diamond)} \neq 0, \tag{11.8}$$

then this solution can be analytically continued to a 2-parameter family (on μ and τ) $\varphi_{c_e}(t; r_0 = r_0^\diamond + O(\mu, \overline{\tau}), \dot{r}_0 = 0, z_0 = 0, \dot{z}_0 = \dot{z}_0^\diamond + O(\mu, \overline{\tau}), \mu)$ of r-symmetric periodic solutions of the reduced isosceles problem, with angular momentum $c = c_e$ and $\mu \geqslant 0$, that have period τ for $(\mu, \overline{\tau} = \tau - \tau^\diamond) \in W_r$, with W_r a sufficiently small neighborhood of $(0,0)$.

11.2.1. Application of Theorem 11.4. Now let $\varphi_{c_e}(t; r_0 = r_0^\diamond = (1 \pm e)/2, \dot{r}_0 = 0, z_0 = 0, \dot{z}_0 = \dot{z}_0^\diamond = \pm\sqrt{2h+4} + O(e), \mu = 0)$ be one of the r-symmetric periodic solutions of the reduced elliptic Sitnikov problem given by Theorem 8.2 for fixed $e > 0$ small and $\tau^\diamond = 2\pi p = qT$ with $p, q \in \mathbb{N}$ coprime and $p > q/(2\sqrt{2})$. By Theorem 11.4, the r-symmetric periodic solution $\varphi_{c_e}(t; r_0^\diamond, 0, 0, \dot{z}_0^\diamond, 0)$ can be continued if (11.8) holds. The value of the derivative (11.8) is obtained from the solution, evaluated at $t = \tau^\diamond/2$, of the variational equations of the elliptic Sitnikov problem (9.3) along the solution curve $(z(t), \dot{z}(t)) = (z(t; r_0^\diamond, \dot{z}_0 = \dot{z}_0^\diamond = \dot{z}_0^* + O(e), 0), \dot{z}(t; r_0^\diamond, \dot{z}_0 = \dot{z}_0^\diamond = \dot{z}_0^* + O(e), 0))$. We have seen that if e is sufficiently small, then the solution of those variational equations is given by the solution of the variational equations of the reduced circular Sitnikov problem along the solution curve $(z_{(0)}(t), \dot{z}_{(0)}(t))$ plus terms of at least order one in e, where $(z_{(0)}(t), \dot{z}_{(0)}(t))$ is the solution of the circular Sitnikov problem with initial conditions $z_{(0)}(0) = 0$, $\dot{z}_{(0)}(0) = \dot{z}_0^*$. Proceeding as in the continuation of doubly symmetric periodic solutions (see section 11.1.1), we can see that

$$\left.\frac{\partial z(\tau/2; r_0, \dot{z}_0, \mu)}{\partial \dot{z}_0}\right|_{(\mu=0,\tau=\tau^\diamond,r_0=r_0^\diamond,\dot{z}_0=\dot{z}_0^\diamond)} = \frac{(-1)^q q}{\sqrt{2}(2k^2-1)^2 k'^2} g(k) + O(e),$$

which is different from zero if the eccentricity e is small enough.

In short, if the eccentricity e is sufficiently small, then $\varphi_{c_e}(t; r_0^\diamond, 0, 0, \dot{z}_0^\diamond, 0)$ can be continued to a family $\varphi_{c_e}(t; r_0 = r_0^\diamond + O(\mu,\overline{\tau}), \dot{r}_0 = 0, z_0 = 0, \dot{z}_0 = \dot{z}_0^\diamond + O(\mu,\overline{\tau}), \mu)$ of r-symmetric periodic solutions of the reduced isosceles problem, with angular momentum $c = c_e$ and $\mu \geqslant 0$, that have period τ for $(\mu, \overline{\tau} = \tau - \tau^\diamond) \in W_r$.

We note that if p is even, then $\varphi_{c_e}(t; r_0^\diamond, 0, 0, \dot{z}_0^\diamond, 0)$ is a doubly symmetric periodic solution. Thus, if the eccentricity e is sufficiently small, then it can also be continued to a family $\varphi_{c_e}(t; r_0 = r_0^\diamond + O(\mu,\overline{\tau}), \dot{r}_0 = 0, z_0 = 0, \dot{z}_0 = \dot{z}_0^\diamond + O(\mu,\overline{\tau}), \mu)$ of doubly symmetric periodic solutions of the reduced isosceles problem, with $c = c_e$ and $\mu \geqslant 0$, that have period τ for $(\mu, \overline{\tau}) \in W_d$ (see Corollary 11.3). Due to the uniqueness of the functions $r_0(\mu,\tau)$ and $\dot{z}_0(\mu,\tau)$ given by the implicit function theorem we have that if p is even and $(\mu,\overline{\tau}) \in W_d \cap W_r$, then the r-symmetric periodic solutions $\varphi_{c_e}(t; r_0^\diamond + O(\mu,\overline{\tau}), 0, 0, \dot{z}_0^\diamond + O(\mu,\overline{\tau}), \mu)$ are doubly symmetric periodic solutions.

If p is odd, then $\varphi_{c_e}(t; r_0^\diamond, 0, 0, \dot{z}_0^\diamond, 0)$ is an r-symmetric periodic solution that is not doubly symmetric because $\dot{r}(\tau^\diamond/4, r_0^\diamond, \dot{z}_0^\diamond, 0) \neq 0$. So, $\dot{r}(\tau/4, r_0^\diamond + O(\mu,\overline{\tau}), \dot{z}_0^\diamond + O(\mu,\overline{\tau}), \mu) \neq 0$ for $(\mu, \overline{\tau} = \tau - \tau^\diamond)$ in a sufficiently small neighborhood of $(0,0)$. Consequently the r-symmetric periodic solutions $\varphi_{c_e}(t; r_0^\diamond + O(\mu,\overline{\tau}), 0, 0, \dot{z}_0^\diamond + O(\mu,\overline{\tau}), \mu)$ are not doubly symmetric periodic solutions. In short, we have proved the following result.

THEOREM 11.5. *For fixed $e > 0$ sufficiently small, let $\varphi_{c_e}(t; r_0 = r_0^\diamond = (1 \pm e)/2, \dot{r}_0 = 0, z_0 = 0, \dot{z}_0 = \dot{z}_0^\diamond = \pm\sqrt{2h+4} + O(e), \mu = 0)$ be one of the r-symmetric periodic solutions of the reduced elliptic Sitnikov problem given by Theorem* 8.2 *that has period $\tau^\diamond = 2\pi p = qT$ for given values of $p, q \in \mathbb{N}$ coprime and $p > q/(2\sqrt{2})$.*

1. *This solution can be continued to a* 2*-parameter family (on μ and τ) $\varphi_{c_e}(t; r_0 = r_0^\diamond + O(\mu,\overline{\tau}), \dot{r}_0 = 0, z_0 = 0, \dot{z}_0 = \dot{z}_0^\diamond + O(\mu,\overline{\tau}), \mu)$ of r-symmetric periodic solutions of the reduced isosceles problem, with angular momentum $c = c_e$ and $\mu \geqslant 0$, that have period τ for $(\mu, \overline{\tau} = \tau - \tau^\diamond)$ in a sufficiently small neighborhood of* $(0,0)$.
2. *If p is odd, then the r-symmetric periodic solutions $\varphi_{c_e}(t; r_0^\diamond + O(\mu,\overline{\tau}), 0, 0, \dot{z}_0^\diamond + O(\mu,\overline{\tau}), \mu)$ are not doubly symmetric, whereas if p is even, they are doubly symmetric.*

11.3. Continuation of *t*-symmetric periodic solutions. By Proposition 5.2, $\varphi_{c_e}(t; r_0, 0, z_0, 0, \mu) = (r(t; r_0, z_0, \mu), \dot{r}(t; r_0, z_0, \mu), z(t; r_0, z_0, \mu), \dot{z}(t; r_0, z_0, \mu))$ is a t-symmetric periodic solution of the reduced isosceles problem (3.1), with $c = c_e$ having period τ, if it satisfies

$$\dot{r}(\tau/2; r_0, z_0, \mu) = 0, \qquad \dot{z}(\tau/2; r_0, z_0, \mu) = 0, \tag{11.9}$$

and $\dot{r}$, $\dot{z}$ are not simultaneously zero for $t \in (0, \tau/2)$.

Let $\varphi_{c_e}(t; r_0 = r_0^\diamond = (1 \pm e)/2, \dot{r}_0 = 0, z_0 = z_0^\diamond, \dot{z}_0 = 0, \mu = 0)$ be a t-symmetric periodic solution of the reduced elliptic Sitnikov problem for a fixed $0 < e < 1$ and let

$\tau^\diamond = 2\pi p$ with $p \in \mathbb{N}$ be its period. That is, let $\tau = \tau^\diamond$, $r_0 = r_0^\diamond$, $z_0 = z_0^\diamond$, and $\mu = 0$ be a solution of system (11.9). Applying the implicit function theorem to system (11.9) in a neighborhood of that solution, choosing μ and τ as the independent variables, if

$$\det \begin{pmatrix} \dfrac{\partial \dot{r}(\tau/2; r_0, z_0, \mu)}{\partial r_0} & \dfrac{\partial \dot{r}(\tau/2; r_0, z_0, \mu)}{\partial z_0} \\ \dfrac{\partial \dot{z}(\tau/2; r_0, z_0, \mu)}{\partial r_0} & \dfrac{\partial \dot{z}(\tau/2; r_0, z_0, \mu)}{\partial z_0} \end{pmatrix}_{\big|(\mu=0,\tau=\tau^\diamond,r_0=r_0^\diamond,z_0=z_0^\diamond)} \neq 0,$$

then for each $(\mu, \overline{\tau} = \tau - \tau^\diamond)$ in a sufficiently small neighborhood W_t of $(0,0)$ with $\mu \geqslant 0$, we can find two unique analytic functions $r_0(\mu,\tau) = r_0^\diamond + O(\mu,\overline{\tau})$ and $z_0(\mu,\tau) = z_0^\diamond + O(\mu,\overline{\tau})$ such that $\varphi_{c_e}(t; r_0^\diamond + O(\mu,\overline{\tau}), 0, z_0^\diamond + O(\mu,\overline{\tau}), 0, \mu)$ is a t-symmetric periodic solution of period τ for the reduced isosceles problem (3.1) for $c = c_e$ and $\mu \geqslant 0$ small enough. The derivatives that appear in this determinant are obtained by evaluating at time $t = \tau^\diamond/2$ the corresponding solutions of the variational equations of the reduced restricted isosceles problem (6.2) for $c = c_e$ along the solution curve $\varphi_{c_e}(t; r_0^\diamond, 0, z_0^\diamond, 0, 0)$. The solution of these variational equations was studied in section 9. Since the first equation of (6.2) does not depend on z and $\dot{z}$, $r(t; r_0, z_0, 0)$ and $\dot{r}(t; r_0, z_0, 0)$ do not depend on the initial conditions $z(0; r_0, z_0, 0)$ and $\dot{z}(0; r_0, z_0, 0)$. So, using the computations made in section 11.2 (see (11.6) and (11.7)) we can prove the following result.

THEOREM 11.6. *Let* $\varphi_{c_e}(t; r_0 = r_0^\diamond = (1 \pm e)/2, \dot{r}_0 = 0, z_0 = z_0^\diamond, \dot{z}_0 = 0, \mu = 0)$ *be a t-symmetric periodic solution of the reduced elliptic Sitnikov problem for a fixed* $0 < e < 1$ *and let* $\tau^\diamond = 2\pi p$ *with* $p \in \mathbb{N}$ *be its period. If*

$$\left.\frac{\partial \dot{z}(\tau/2; r_0, z_0, \mu)}{\partial z_0}\right|_{(\mu=0,\tau=\tau^\diamond,r_0=r_0^\diamond,z_0=z_0^\diamond)} \neq 0, \tag{11.10}$$

then this solution can be analytically continued to a 2*-parameter family (on* μ *and* τ*)* $\varphi_{c_e}(t; r_0 = r_0^\diamond + O(\mu,\overline{\tau}), \dot{r}_0 = 0, z_0 = z_0^\diamond + O(\mu,\overline{\tau}), \dot{z}_0 = 0, \mu)$ *of t-symmetric periodic solutions of the reduced isosceles problem, with angular momentum* $c = c_e$ *and* $\mu \geqslant 0$*, that have period* τ *for* $(\mu, \overline{\tau} = \tau - \tau^\diamond) \in W_t$*, with* W_t *a sufficiently small neighborhood of* $(0,0)$.

11.3.1. Application of Theorem 11.6. Now we apply Theorem 11.6 to continue the t-symmetric periodic solutions of the reduced elliptic Sitnikov problem given by Theorem 8.3. Let $\varphi_{c_e}(t; r_0 = r_0^\diamond = (1 \pm e)/2, \dot{r}_0 = 0, z_0 = z_0^\diamond = z_0^* + O(e), \dot{z}_0 = 0, \mu = 0)$, with $z_0^* = \pm\sqrt{\frac{1}{h^2} - \frac{1}{4}}$, be one of these periodic solutions for a fixed $e > 0$ sufficiently small and fixed $p, q \in \mathbb{N}$ coprime with $p > q/(2\sqrt{2})$. The t-symmetric periodic solution $\varphi_{c_e}(t; r_0^\diamond, 0, z_0^\diamond, 0, 0)$ can be continued if (11.10) holds. Proceeding in a similar way to that of sections 11.1 and 11.2, we can see that if the eccentricity e is sufficiently small, then

$$\left.\frac{\partial \dot{z}(\tau/2; r_0, \dot{z}_0, \mu)}{\partial z_0}\right|_{(\mu=0,\tau=\tau^\diamond,r_0=r_0^\diamond,z_0=z_0^\diamond)} = (-1)^{q+1} 4\sqrt{2}(1-2k^2)^2 q\, g(k) + O(e) \neq 0.$$

In short, if the eccentricity e is sufficiently small, then $\varphi_{c_e}(t; r_0^\diamond, 0, z_0^\diamond, 0, 0)$ can be continued to a family $\varphi_{c_e}(t; r_0 = r_0^\diamond + O(\mu,\overline{\tau}), \dot{r}_0 = 0, z_0 = z_0^\diamond + O(\mu,\overline{\tau}), \dot{z}_0 = 0, \mu)$ of t-symmetric periodic solutions of the reduced isosceles problem, with angular momentum $c = c_e$ and $\mu \geqslant 0$, that have period τ for $(\mu, \overline{\tau} = \tau - \tau^\diamond) \in W_t$. Moreover, due to the uniqueness of the functions $r_0(\mu,\tau)$ and $z_0(\mu,\tau)$ given by the implicit function theorem we can see that the t-symmetric periodic solutions $\varphi_{c_e}(t; r_0^\diamond + O(\mu, \overline{\tau}), 0, z_0^\diamond + O(\mu, \overline{\tau}), 0, \mu)$ are doubly symmetric when p is even, and they are not doubly symmetric when p is odd (see the arguments of section 11.2.1). Therefore we have proved the following result.

THEOREM 11.7. *For fixed $e > 0$ sufficiently small, let $\varphi_{c_e}(t; r_0 = r_0^\diamond = (1 \pm e)/2, \dot{r}_0 = 0, z_0 = z_0^\diamond = z_0^* + O(e), \dot{z}_0 = 0, \mu = 0)$, with $z_0^* = \pm\sqrt{\frac{1}{h^2} - \frac{1}{4}}$, be one of the t-symmetric periodic solutions of the reduced elliptic Sitnikov problem given by Theorem 8.3 that has period $\tau^\diamond = 2\pi p = qT$ for given values of $p, q \in \mathbb{N}$ coprime and $p > q/(2\sqrt{2})$.*

1. *This solution can be continued to a 2-parameter family (on μ and τ) $\varphi_{c_e}(t; r_0 = r_0^\diamond + O(\mu, \overline{\tau}), \dot{r}_0 = 0, z_0 = z_0^\diamond + O(\mu, \overline{\tau}), \dot{z}_0 = 0, \mu)$ of t-symmetric periodic solutions of the reduced isosceles problem, with angular momentum $c = c_e$ and $\mu \geqslant 0$, that have period τ for $(\mu, \overline{\tau} = \tau - \tau^\diamond)$ in a sufficiently small neighborhood of $(0, 0)$.*
2. *If p is odd, then the t-symmetric periodic solutions $\varphi_{c_e}(t; r_0^\diamond + O(\mu, \overline{\tau}), 0, z_0^\diamond + O(\mu, \overline{\tau}), 0, \mu)$ are not doubly symmetric, whereas if p is even, they are doubly symmetric.*

11.4. Remarks. In Theorems 11.5 and 11.7, we continued eight periodic solutions of the reduced elliptic Sitnikov problem: $\varphi_{c_e}(t; r_0 = r_0^\diamond = \frac{(1\pm e)}{2}, \dot{r}_0 = 0, z_0 = 0, \dot{z}_0 = \pm\sqrt{2h+4} + O(e), \mu = 0)$ and $\varphi_{c_e}(t; r_0 = r_0^\diamond = \frac{(1\pm e)}{2}, \dot{r}_0 = 0, z_0 = \pm\sqrt{\frac{1}{h^2} - \frac{1}{4}} + O(e), \dot{z}_0 = 0, \mu = 0)$. But not all eight periodic solutions give different periodic orbits of the reduced elliptic Sitnikov problem (see Theorem 8.4). Since the reduced isosceles problem is an autonomous system, if we continue different periodic solutions that define the same periodic orbit, then we will obtain the same periodic orbit of the reduced isosceles problem. Therefore, Corollary 11.3 and Theorems 11.5 and 11.7 prove Theorem 11.1.

We note that in order to continue the symmetric periodic solutions of the reduced elliptic Sitnikov problem, we applied the implicit function theorem, choosing μ and τ as the independent variables. As happened in the continuation of periodic solutions from the reduced circular Sitnikov problem (see section 10), there are other possible choices for the independent variables. These other possible choices are (μ, r_0), $(\mu, \dot{z}_0)$, and $(\mu, \widetilde{h})$ (respectively, (μ, r_0), (μ, z_0), $(\mu, \widetilde{h})$) when the starting initial condition that we continue is r-symmetric (respectively, t-symmetric). Here $\widetilde{h}$ is the energy of the periodic solution. We have analyzed these choices for the independent variables, but we have not obtained new periodic orbits. In particular, we have seen that the determinant that we must evaluate when we use $(\mu, \dot{z}_0)$ (respectively, (μ, z_0)) as the independent variables is more complicated than in the other cases because we do not know an explicit expression of some of the derivatives.

In particular, we also have analyzed the continuation of the symmetric periodic solutions of the reduced elliptic Sitnikov problem given by Theorems 8.2 and 8.3 to symmetric periodic solutions of the reduced isosceles problem by fixing either the period, one of the initial conditions, or the energy. We have seen that if the eccentricity e is sufficiently small, then these symmetric periodic solutions can be continued to families of symmetric periodic solutions of the reduced isosceles problem for $\mu > 0$ sufficiently small that have either the same period, the same initial condition r_0, or the same energy $\widetilde{h}$ as the initial orbit. We have also evaluated numerically for some periodic orbits the correspondent determinant when we continue by fixing the initial condition $\dot{z}_0$ (respectively, z_0), and we have seen that it is different from zero.

We note that in order to apply successfully the implicit function theorem it is very important to choose a good set of independent variables.

12. From reduced circular Sitnikov problem to reduced isosceles problem in two steps. In section 10 we have continued directly the periodic orbits of the reduced circular Sitnikov problem with period $T \neq 4\pi n$ for all $n \in \mathbb{N}$ to doubly symmetric periodic orbits of the reduced isosceles problem for $\mu > 0$ sufficiently small having period near T and fixed angular momentum $c = 1/4$. Now we continue, by using two steps, the periodic orbits of the reduced circular Sitnikov problem with rational period $T = 2\pi p/q$ for all $p, q \in \mathbb{N}$ coprime and $p > q/(2\sqrt{2})$ to symmetric periodic orbits of the reduced isosceles problem for $\mu > 0$ sufficiently small having period near $2\pi p$ and fixed angular momentum $c = 1/4$. First, we continue them to periodic orbits of the reduced elliptic Sitnikov problem for sufficiently small values of e, and then we continue the periodic orbits of the reduced elliptic Sitnikov problem to the reduced isosceles problem for $\mu > 0$ sufficiently small, always having fixed angular momentum $c = 1/4$. The main differences between direct continuation and continuation in two steps are analyzed at the end of this section.

LEMMA 12.1. *Let $\varphi(t) = (r(t), \dot r(t), z(t), \dot z(t))$ be a periodic solution of the reduced isosceles problem* (3.1) *with $c = c_e$ having initial conditions $r(0) = r_0$, $\dot r(0) = 0$, $z(0) = z_0$, $\dot z(0) = \dot z_0$ and period τ. If we set $\alpha = 1/(1-e^2)$, $\widetilde r(t) = \alpha r(\alpha^{3/2} t)$, $\dot{\widetilde r}(t) = \alpha^{-1/2} \dot r(\alpha^{3/2} t)$, $\widetilde z(t) = \alpha z(\alpha^{3/2} t)$, and $\dot{\widetilde z}(t) = \alpha^{-1/2} \dot z(\alpha^{3/2} t)$, then $\gamma(t) = (\widetilde r(t), \dot{\widetilde r}(t), \widetilde z(t), \dot{\widetilde z}(t))$ is a periodic solution of the reduced isosceles problem* (3.1) *with $c = 1/4$ having initial conditions $\widetilde r(0) = \alpha r_0$, $\dot{\widetilde r}(0) = 0$, $\widetilde z(0) = \alpha z_0$, $\dot{\widetilde z}(0) = \alpha^{-1/2} \dot z_0$ and period $\widetilde\tau = \alpha^{-3/2}\tau$.*

Proof. The proof is an immediate consequence of Proposition 3.1. ∎

Remark 12.2. We note that the period $\widetilde\tau = \widetilde\tau(e) = \tau(1-e^2)^{3/2}$ is a decreasing function in $(0,1)$, so in this interval the function $\widetilde\tau(e)$ has the inverse

$$e(\widetilde\tau) = \sqrt{1 - \left(\frac{\widetilde\tau}{\tau}\right)^{2/3}}.$$

Therefore, the solution $\gamma(t) = (\widetilde r(t), \dot{\widetilde r}(t), \widetilde z(t), \dot{\widetilde z}(t))$ can be parameterized by the period $\widetilde\tau$ instead of the eccentricity e.

Let γ_{pq} be the periodic orbit of the reduced circular Sitnikov problem with period $T = 2\pi p/q$ for given $p, q \in \mathbb{N}$ coprime with $p > q/(2\sqrt{2})$. Choosing conveniently the origin of time, γ_{pq} can be thought of as the orbit associated to either the solutions $\varphi_{1/4}\big(t; r_0 = 1/2, \dot r_0 = 0, z_0 = 0, \dot z_0 = \dot z_0^* = \pm\sqrt{2h+4}, \mu = 0\big)$ or the solutions $\varphi_{1/4}(t; r_0 = 1/2, \dot r_0 = 0, z_0 = z_0^* = \pm\sqrt{\frac{1}{h^2} - \frac14}, \dot z_0 = 0, \mu = 0)$, where h is such that $T = T(h) = 2\pi p/q$.

We start analyzing the continuation in two steps of the periodic solutions $\varphi_{1/4}(t; 1/2, 0, 0, \dot z_0^* = \pm\sqrt{2h+4}, 0\big)$ to r-symmetric periodic solutions of the reduced isosceles problem with $c = 1/4$ and $\mu > 0$ sufficiently small. Afterward we will analyze the continuation in two steps to t-symmetric periodic solutions of the periodic solutions $\varphi_{1/4}(t; 1/2, 0, z_0^* = \pm\sqrt{\frac{1}{h^2} - \frac14}, 0, 0)$. We note that it is not necessary to consider the continuation in two steps of the above periodic solutions to doubly symmetric periodic solutions, because it can be obtained from the continuation of either r- or t-symmetric periodic solutions having period $T = 2\pi p/q$ with p even.

By Theorem 8.2, each periodic solution $\varphi_{1/4}(t; 1/2, 0, 0, \dot z_0^* = \pm\sqrt{2h+4}, 0)$ can be continued to two families $\varphi_{c_e}(t; r_0 = r_0^P = (1-e)/2, \dot r_0 = 0, z_0 = 0, \dot z_0 = \dot z_0^P = \dot z_0^* +$

$O(e), \mu = 0)$ and $\varphi_{c_e}(t; r_0 = r_0^A = (1+e)/2, \dot r_0 = 0, z_0 = 0, \dot z_0 = \dot z_0^A = \dot z_0^* + O(e), \mu = 0)$ of r-symmetric periodic solutions of the reduced elliptic Sitnikov problem having period $\tau^\diamond = 2\pi p = qT$ for $e > 0$ sufficiently small. Moreover these two families are formed by doubly symmetric periodic solutions if p is even, and they are formed by r-symmetric periodic solutions that are not doubly symmetric if p is odd. Then, using Lemma 12.1, Theorem 8.2 can be stated as follows.

THEOREM 12.3 (reformulation of Theorem 8.2). *Let $\varphi_{1/4}\big(t; r_0 = 1/2, \dot r_0 = 0, z_0 = 0, \dot z_0 = \dot z_0^* = \pm\sqrt{2h+4}, \mu = 0\big)$ be a periodic solution of the reduced circular Sitnikov problem with period $T = 2\pi p/q$ for given $p, q \in \mathbb{N}$ coprime and $p > q/(2\sqrt{2})$. We denote*

$$\widetilde r_0^{P}(e) = \frac{r_0^P}{1-e^2} = \frac{1}{2(1+e)}\ , \qquad \widetilde r_0^{A}(e) = \frac{r_0^A}{1-e^2} = \frac{1}{2(1-e)}\ ,$$
$$\dot{\widetilde z}_0^{P}(e) = \sqrt{1-e^2}\, \dot z_0^P\ , \qquad \dot{\widetilde z}_0^{A}(e) = \sqrt{1-e^2}\, \dot z_0^A\ .$$

1. *The solution $\varphi_{1/4}(t; 1/2, 0, 0, \dot z_0^*, 0)$ can be continued to two families $\varphi_{1/4}(t; r_0 = \widetilde r_0^P(e), \dot r_0 = 0, z_0 = 0, \dot z_0 = \dot{\widetilde z}_0^P(e), \mu = 0)$ and $\varphi_{1/4}(t; r_0 = \widetilde r_0^A(e), \dot r_0 = 0, z_0 = 0, \dot z_0 = \dot{\widetilde z}_0^A(e), \mu = 0)$ of r-symmetric periodic solutions of the reduced elliptic restricted isosceles problem with angular momentum $c = 1/4$ having period $\widetilde\tau = 2\pi p(1-e^2)^{3/2}$ for $e \in (0, \overline e)$ with $\overline e$ sufficiently small.*
2. *If p is odd, the r-symmetric periodic solutions $\varphi_{1/4}(t; \widetilde r_0^{P,\,A}(e), 0, 0, \dot{\widetilde z}_0^{P,\,A}(e), 0)$ are not doubly symmetric, whereas if p is even, then they are doubly symmetric.*

Let $\varphi_{c_e}(t; r_0 = r_0^\diamond, \dot r_0 = 0, z_0 = 0, \dot z_0 = \dot z_0^\diamond, \mu = 0)$ be one of the r-symmetric periodic solutions of the reduced elliptic Sitnikov problem given by Theorem 8.2 for fixed values of p, q and $e > 0$ small. If e is sufficiently small, then from Theorem 11.5, this r-symmetric periodic solution can be continued to a 2-parameter family (on μ and τ) $\varphi_{c_e}(t; r_0 = r_0(\mu,\tau), \dot r_0 = 0, z_0 = 0, \dot z_0 = \dot z_0(\mu,\tau), \mu)$ of r-symmetric periodic solutions of the reduced isosceles problem, with angular momentum $c = c_e$ and $\mu \geqslant 0$, that have period τ for (μ,τ) in a sufficiently small neighborhood W of $(0,\tau^\diamond)$. Moreover $r_0(\mu,\tau)$ and $\dot z_0(\mu,\tau)$ are the two unique analytic functions defined in W such that $r_0(0,\tau^\diamond) = r_0^\diamond$ and $\dot z_0(0,\tau^\diamond) = \dot z_0^\diamond$. We note that, by Lemma 12.1,

$$\varphi_{1/4}\left(t; r_0 = \overline r_0 = \frac{r_0(\mu,\tau)}{1-e^2}, \dot r_0 = 0, z_0 = 0, \dot z_0 = \dot{\overline z}_0 = \sqrt{1-e^2}\, \dot z_0(\mu,\tau), \mu\right)$$

is an r-symmetric periodic solution of the reduced isosceles problem, with angular momentum $c = 1/4$ and $\mu \geqslant 0$, that has period $\overline\tau = \tau(1-e^2)^{3/2}$. In short, Theorem 11.5 can be stated as follows.

THEOREM 12.4 (reformulation of Theorem 11.5). *Let $\varphi_{1/4}(t; r_0 = \widetilde r_0^{P,\,A}, \dot r_0 = 0, z_0 = 0, \dot z_0 = \dot{\widetilde z}_0^{P,\,A}, \mu = 0)$ be one of the r-symmetric periodic solutions of the reduced elliptic restricted isosceles problem given by Theorem 12.3 for fixed $e > 0$ sufficiently small and $p, q \in \mathbb{N}$ coprime with $p > q/(2\sqrt{2})$.*

1. *This solution can be continued to a 2-parameter family (on μ and $\overline\tau$) $\varphi_{1/4}\big(t;\ r_0 = \overline r_0(\mu,\overline\tau), \dot r_0 = 0, z_0 = 0, \dot z_0 = \dot{\overline z}_0(\mu,\overline\tau), \mu\big)$ of r-symmetric periodic solutions of the reduced isosceles problem, with angular momentum $c = 1/4$ and $\mu \geqslant 0$, that have period $\overline\tau$ for $(\mu,\overline\tau)$ in a sufficiently small neighborhood $\overline W$ of $\big(0, 2\pi p(1-e^2)^{3/2}\big)$. Moreover $\overline r_0(\mu,\overline\tau)$ and $\dot{\overline z}_0(\mu,\overline\tau)$ are the two unique analytic functions defined in $\overline W$ such that $\overline r_0\left(0, 2\pi p(1-e^2)^{3/2}\right) = \widetilde r_0^{P,\,A}$ and $\dot{\overline z}_0\left(0, 2\pi p(1-e^2)^{3/2}\right) = \dot{\widetilde z}_0^{P,\,A}$.*

2. *If p is odd, the r-symmetric periodic solutions $\varphi_{c_e}(t;\overline{r}_0(\mu,\overline{\tau}),0,0,\dot{\overline{z}}_0(\mu,\overline{\tau}),\mu)$ are not doubly symmetric, whereas if p is even, they are doubly symmetric.*

We note that using Remark 12.2, the solutions obtained from Theorems 12.3 and 12.4 can be parameterized by means of the period $\widetilde{\tau}$ and $\overline{\tau}$, respectively, instead of the eccentricity.

Using the period instead of the eccentricity as a parameter, the r-symmetric periodic solutions of the reduced restricted isosceles problem $\varphi_{1/4}(t;\widetilde{r}_0^{P,A}(e),0,0,\dot{\widetilde{z}}_0^{P,A}(e),0)$ given by Theorem 12.3 become $\varphi_{1/4}(t;\widehat{r}_0^{P,A}(\widetilde{\tau}),0,0,\dot{\widehat{z}}_0^{P,A}(\widetilde{\tau}),0)$, where $\widehat{r}_0^{P,A}(\widetilde{\tau}) = \widetilde{r}_0^{P,A}(e(\widetilde{\tau}))$ and $\dot{\widehat{z}}_0^{P,A}(\widetilde{\tau}) = \dot{\widetilde{z}}_0^{P,A}(e(\widetilde{\tau}))$, with

$$e(\widetilde{\tau}) = \sqrt{1-\left(\frac{\widetilde{\tau}}{2\pi p}\right)^{2/3}}$$

and $\widetilde{\tau} \in (\widetilde{\tau}_1,\widetilde{\tau}_2) = \left(\tau^{\diamond}(1-\overline{e}^2)^{3/2},\tau^{\diamond}\right)$ for $\overline{e}$ sufficiently small. On the other hand, from Theorem 12.4, we have that, for a fixed value of $\widetilde{\tau}^* \in (\widetilde{\tau}_1,\widetilde{\tau}_2)$, we can find two unique analytic functions $\overline{r}_0^{P,A}(\mu,\overline{\tau})$ and $\dot{\overline{z}}_0^{P,A}(\mu,\overline{\tau})$ in such a way that $\varphi_{1/4}(t;r_0 = \overline{r}_0^{P,A}(\mu,\overline{\tau}),\dot{r}_0=0,z_0=0,\dot{z}_0=\dot{\overline{z}}_0^{P,A}(\mu,\overline{\tau}),\mu)$ is an r-symmetric periodic solution of the reduced isosceles problem, with angular momentum $c=1/4$ and $\mu \geqslant 0$, that has period $\overline{\tau}$ for $(\mu,\overline{\tau})$ in a sufficiently small neighborhood $\overline{W}$ of $(0,\widetilde{\tau}^*)$. Moreover $\overline{r}_0^{P,A}(\mu,\overline{\tau})$ and $\dot{\overline{z}}_0^{P,A}(\mu,\overline{\tau})$ are the two unique analytic functions defined in $\overline{W}$ such that $\overline{r}_0^{P,A}(0,\widetilde{\tau}^*) = \widehat{r}_0^{P,A}(\widetilde{\tau}^*)$ and $\dot{\overline{z}}_0^{P,A}(0,\widetilde{\tau}^*) = \dot{\widehat{z}}_0^{P,A}(\widetilde{\tau}^*)$. In particular, $\overline{r}_0^{P,A}(0,\widetilde{\tau}) = \widehat{r}_0^{P,A}(\widetilde{\tau})$ and $\dot{\overline{z}}_0^{P,A}(0,\widetilde{\tau}) = \dot{\widehat{z}}_0^{P,A}(\widetilde{\tau})$ for all $(0,\widetilde{\tau}) \in \overline{W}$. Then using the compactness argument of Theorem 10.2 and working again with the parameter e instead of $\widetilde{\tau}$, Theorem 12.4 can be improved as follows.

THEOREM 12.5. *For fixed $p,q\in\mathbb{N}$ coprime with $p > q/(2\sqrt{2})$, for any interval $[e_1,e_2]$ with $0<e_1<e_2<\overline{e}$ and $\overline{e}$ sufficiently small, we can find $\mu_0>0$ and analytic functions $r_0^P(\mu,e)$, $\dot{z}_0^P(\mu,e)$, $r_0^A(\mu,e)$, $\dot{z}_0^A(\mu,e)$ defined for all $\mu\in[0,\mu_0)$ and $e\in[e_1,e_2]$ such that $\varphi_{1/4}(t;r_0^P(\mu,e),0,0,\dot{z}_0^P(\mu,e),\mu)$ and $\varphi_{1/4}(t;r_0^A(\mu,e),0,0,\dot{z}_0^A(\mu,e),\mu)$ are r-symmetric periodic solutions of the reduced isosceles problem* (3.1)*, with angular momentum $c=1/4$, that have period $\overline{\tau}=2\pi p(1-e^2)^{3/2}$. Moreover*

$$r_0^P(0,e) = \frac{1}{2(1+e)}\ , \quad \dot{z}_0^P(0,e) = \dot{\widetilde{z}}_0^P(e)\ , \quad r_0^A(0,e) = \frac{1}{2(1-e)}\ , \quad \dot{z}_0^A(0,e) = \dot{\widetilde{z}}_0^A(e)\ ,$$

where the functions $\dot{\widetilde{z}}_0^P(e)$ and $\dot{\widetilde{z}}_0^A(e)$ are the ones given by Theorem 12.3.

Moreover if p is even, then the continued periodic solutions are doubly symmetric, whereas if p is odd, then they are r- but not doubly symmetric.

In short, from Theorems 12.3 and 12.5, we have the following result.

THEOREM 12.6. *The two periodic solutions of the reduced circular Sitnikov problem $\varphi_{1/4}\left(t;1/2,0,0,\pm\sqrt{2h+4},0\right)$ having period $T=2\pi p/q$ for given $p,q\in\mathbb{N}$ coprime with $p>q/(2\sqrt{2})$ can be continued by two steps to two 2-parameter families (on μ and e) $\varphi_{1/4}(t;r_0^P(\mu,e),0,0,\dot{z}_0^P(\mu,e),\mu)$ and $\varphi_{1/4}(t;r_0^A(\mu,e),0,0,\dot{z}_0^A(\mu,e),\mu)$ of r-symmetric periodic solutions of the reduced isosceles problem* (3.1)*, with angular momentum $c=1/4$ and $\mu\geqslant 0$ sufficiently small, that have period $\overline{\tau}=2\pi p(1-e^2)^{3/2}$ for $e>0$ sufficiently small. Furthermore if p is even, then the continued periodic solutions are doubly symmetric, whereas if p is odd, then they are r- but not doubly symmetric.*

Applying to the t-symmetric periodic solutions $\varphi_{1/4}(t;1/2,0,z_0^*=\pm\sqrt{\frac{1}{h^2}-\frac{1}{4}},0,0)$

the arguments that we have used to continue the r-symmetric periodic solutions in two steps, we obtain the following result.

THEOREM 12.7. *The two periodic solutions of the reduced circular Sitnikov problem* $\varphi_{1/4}(t;1/2,0,\pm\sqrt{\frac{1}{h^2}-\frac{1}{4}},0,0)$ *having period* $T=2\pi p/q$ *for given* $p,q\in\mathbb{N}$ *coprime with* $p>q/(2\sqrt{2})$ *can be continued by two steps to two* 2*-parameter families (on* μ *and* e*)* $\varphi_{1/4}\,(t;r_0^P(\mu,e),0,z_0^P(\mu,e),0,\mu)$ *and* $\varphi_{1/4}\,(t;r_0^A(\mu,e),0,\ z_0^A(\mu,e),0,\mu)$ *of* t*-symmetric periodic solutions of the reduced isosceles problem* (3.1)*, with angular momentum* $c=1/4$ *and* $\mu\geqslant 0$ *sufficiently small, that have period* $\overline{\tau}=2\pi p(1-e^2)^{3/2}$ *for* $e>0$ *sufficiently small. Furthermore if* p *is even, then the continued periodic solutions are doubly symmetric, whereas if* p *is odd, then they are* t*-symmetric but not doubly symmetric.*

By Theorems 12.6 and 12.7 the periodic orbit of the reduced circular Sitnikov problem with period $T=2\pi p/q$ for given $p,q\in\mathbb{N}$ coprime and $p>q/(2\sqrt{2})$ can be continued in two steps to eight 2-parameter families (on μ and e) of symmetric periodic orbits of the reduced isosceles problem (3.1) with angular momentum $c=1/4$ and $\mu\geqslant 0$ small. But not all eight families of symmetric periodic orbits are different.

THEOREM 12.8. *Let* γ_{pq} *be the periodic orbit of the reduced circular Sitnikov problem with period* $T=2\pi p/q$ *for given* $p,q\in\mathbb{N}$ *coprime with* $p>q/(2\sqrt{2})$*.*

1. *If* p *is odd, then* γ_{pq} *can be continued by two steps to four* 2*-parameter families (on* μ *and* e*) of symmetric periodic orbits of the reduced isosceles problem* (3.1)*, with angular momentum* $c=1/4$ *and* $\mu\geqslant 0$ *sufficiently small, that have period* $\tau=2\pi p(1-e^2)^{3/2}$ *with* $e>0$ *sufficiently small. Moreover, two of these families are formed by* r*-symmetric periodic orbits that are not doubly symmetric, and the other two are formed by* t*-symmetric periodic orbits that are not doubly symmetric.*
2. *If* p *is even, then* γ_{pq} *can be continued by two steps to two* 2*-parameter families (on* μ *and* e*) of doubly symmetric periodic orbits of the reduced isosceles problem* (3.1)*, with angular momentum* $c=1/4$ *and* $\mu\geqslant 0$ *sufficiently small, that have period* $\tau=2\pi p(1-e^2)^{3/2}$ *with* $e>0$ *sufficiently small.*

Proof. From Lemma 12.1, we can see easily that different periodic orbits of the reduced isosceles problem with $c=c_e$ correspond to different periodic orbits of the reduced isosceles problem with $c=1/4$. Thus the proof follows immediately from Theorems 8.4, 11.1, 12.6, and 12.7. ∎

We remark that the periodic orbits γ_{p1} of the reduced circular Sitnikov problem with period $T=2\pi p$ for some even $p\in\mathbb{N}$ cannot be continued by direct continuation. They can only be continued by using two steps. The periodic orbits γ_{pq} with $q\neq 1$ and the ones with p odd and $q=1$ can be continued in both ways, that is, using direct continuation and using continuation in two steps. We note that if we use direct continuation, then γ_{pq} can be continued to a family of doubly symmetric periodic orbits with period near $T=2\pi p/q$. On the other hand, using continuation in two steps, γ_{pq} can be continued to two or four families of symmetric periodic orbits with period near $\tau^\diamond=2\pi p=qT$ (two families of doubly symmetric periodic orbits when p is even, and two families of r-symmetric plus two families of t-symmetric periodic orbits that are not doubly symmetric when p is odd). Therefore if $q\neq 1$, then the periodic orbits obtained from direct continuation and those obtained from continuation in two steps are always different, because they have different periods. Moreover, when p is odd, the orbits obtained from direct continuation are doubly symmetric, whereas the ones obtained from continuation in two steps are r- and t-symmetric, but not doubly symmetric. Therefore when p is odd and $q=1$ the direct continuation and the continuation in two steps also give different periodic orbits. Finally the periodic orbits of the reduced circular Sitnikov problem with period $T = 2\pi\omega$, where $\omega > 1/(2\sqrt{2})$ is an irrational number, can be continued by direct continuation, but they cannot be continued in two steps.

13. Summary. The main results about continuation of the periodic orbits of the reduced circular Sitnikov problem to symmetric periodic orbits of the reduced isosceles problem for $\mu > 0$ sufficiently small—that is, Theorem 10.1 and Theorem 12.8—are summarized in Theorem A of the introduction.

In Remark 12.2 we have seen that we can work with the parameter $\tau = 2\pi p\, f(e)$ (the period) instead of the eccentricity e. Thus the 2-parameter families of periodic orbits of the reduced isosceles problem obtained from continuation in two steps of periodic orbits of the reduced circular Sitnikov problem with period $T = 2\pi p/q$ for $p, q \in \mathbb{N}$ coprime with $p > q/(2\sqrt{2})$ can be parameterized by means of μ and τ instead of μ and e. This means that Theorem A of the introduction can be stated using μ and τ as parameters instead of μ and e.

Next we give the extension of Theorem A to the full isosceles problem (see section 4 for more details about the relationship between the periodic orbits of the reduced isosceles problem and the orbits of the full isosceles problem).

Let Π_T denote the two-dimensional invariant torus of the restricted isosceles problem that comes from a periodic orbit of the reduced circular Sitnikov problem with period T. Then we have the following result.

THEOREM 13.1. *The torus of the circular restricted isosceles problem Π_T with $T > \pi/\sqrt{2}$ can be continued to the following families of two-dimensional tori of the isosceles problem with $\mu > 0$ sufficiently small. These tori are filled with either periodic or quasi-periodic orbits:*

1. *Case $T = 2\pi\omega$ with $\omega > 1/(2\sqrt{2})$ an irrational number.*
   (a) *Π_T can be continued directly to one 2-parameter family (on μ and τ with τ sufficiently close to T) of two-dimensional tori.*
2. *Case $T = 2\pi p/q$ for some $p, q \in \mathbb{N}$ coprime with $p > q/(2\sqrt{2})$.*
   (a) *p odd:*
      i. *Π_T can be continued directly to one 2-parameter family (on μ and τ with τ sufficiently close to T) of two-dimensional tori.*
      ii. *Π_T can be continued by two steps to four 2-parameter families (on μ and τ with τ sufficiently close to Tq) of two-dimensional tori.*
   (b) *p even and $q \neq 1$:*
      i. *Π_T can be continued directly to one 2-parameter family (on μ and τ with τ sufficiently close to T) of two-dimensional tori.*
      ii. *Π_T can be continued by two steps to two 2-parameter families (on μ and τ with τ sufficiently close to Tq) of two-dimensional tori.*
   (c) *p even and $q = 1$:*
      i. *Π_T can be continued by two steps to two 2-parameter families (on μ and τ with τ sufficiently close to Tq) of two-dimensional tori.*

By Proposition 7.7, the tori Π_T are filled with periodic orbits when $T = p2\pi/q$ for some $p, q \in \mathbb{N}$ coprime with $p > q/(2\sqrt{2})$; and they are filled with quasi-periodic orbits when $T = 2\pi\omega$ with $\omega > 1/(2\sqrt{2})$ an irrational number. So, in particular, we have continued tori filled with quasi-periodic orbits. The tori of the isosceles problem for $\mu > 0$ that we have obtained are filled with either periodic or quasi-periodic orbits of the isosceles problem.

Remember that the phase portrait of the isosceles problem on each angular momentum level c with $c \neq 0$ is the same (see Proposition 3.1). Therefore we have obtained invariant periodic and quasi-periodic two-dimensional tori on each angular momentum level $c \neq 0$.

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