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THE NUMBER OF PLANAR CENTRAL CONFIGURATIONS FOR THE 4–BODY PROBLEM IS FINITE WHEN 3 MASS POSITIONS ARE FIXED

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Abstract. In the n–body problem a central configuration is formed when the position vector of each particle with respect to the center of mass is a common scalar multiple of its acceleration vector. Lindstrom showed for $n = 3$ and for $n > 4$ that if $n - 1$ masses are located at fixed points in the plane, then there are only a finite number of ways to position the remaining nth mass in such a way that they define a central configuration. Lindstrom leaves open the case $n = 4$. In this paper we prove the case $n = 4$ using as variables the mutual distances between the particles.

1. Introduction

For the n–body problem a configuration of the system of n particles is central if the acceleration of each mass is proportional to its position relative to the center of mass of the system.

Central configurations play an important role in the n–body problem of celestial mechanics. For instance, they allow one to obtain the homographic solutions (the unique solutions of the n–body problem that we can describe explicitly) [13], central configurations play a main role in the topological changes of the integral manifolds [11], and they are the limiting configurations for colliding particles [7] or parabolic escape [10].

Some interesting results for the planar central configurations of the n–body problem have been achieved, but the problem is far from solved. The sixth problem of Smale's list presenting his challenging mathematical problems for the twenty–first century [12], cites Wintner's question of whether, for a given set of n positive masses, the number of nonequivalent (modulus rotations and rescalings) planar central configurations is finite.

In [5] Lindstrom formulated a program of research as follows: Given n positive masses $m_1, m_2, \ldots, m_n$ and for any $k = 1, 2, \ldots, n-2$, given their $n-k$ positions in the plane, to determine whether there are only a finite number of ways to position the remaining k particles in a manner that defines a central configuration. For given

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n this is a sequence of questions for which $k = n - 2$ is equivalent to the finiteness question of central configurations. Of course, following Lindstrom we assume that the center of mass is unknown.

Lindstrom approaches the case $k = 1$ for any n, leaving open the question for $n = 4$. The goal of this paper is to prove Lindstrom's remaining case for $n = 4$; that is, we prove the next result.

Theorem. *For three given masses m_1, m_2 and m_3 at fixed positions there are only a finite number of different positions in the plane for a given mass m_4 in order to have a central configuration of the planar 4–body problem.*

Using ideas of Dziobek (see [2] or [6]) we formulate the equations for the central configurations of the 4–body problem in the plane as a system of 6 equations using the mutual distances between particles as variables; see for more details Hagihara [3]. After some computations, we write the equations of central configurations as a polynomial system. Then, we use the Bézout Theorem and the theory of resultants to show that, having fixed the four masses and the positions of the first three particles, then there exist a finite number of positions (possibly zero) for the fourth particle.

The paper is organized as follows. In Section 2 we present the system of equations for the central configurations. Our main tools, the resultant of two polynomials and the Bézout Theorem, are introduced in Section 3. Finally, in Section 4 we prove the theorem.

2. Equations for the central configurations

We do not need to study the collinear central configurations of the 4–body problem, because they are well known (see Moulton [8]), and modulo homotheties and rotations there are exactly 12.

The equations for the planar noncollinear central configurations of the 4–body problem with positive masses m_i, for $i = 1, \ldots, 4$, can be written as

$$\begin{array}{c} m_3\Delta_4(r_{13}^{-3} - r_{23}^{-3}) = m_4\Delta_3(r_{14}^{-3} - r_{24}^{-3}), \\ m_2\Delta_4(r_{12}^{-3} - r_{23}^{-3}) = m_4\Delta_2(r_{14}^{-3} - r_{34}^{-3}), \\ m_2\Delta_3(r_{12}^{-3} - r_{24}^{-3}) = m_3\Delta_2(r_{13}^{-3} - r_{34}^{-3}), \\ m_1\Delta_4(r_{12}^{-3} - r_{13}^{-3}) = m_4\Delta_1(r_{24}^{-3} - r_{34}^{-3}), \\ m_1\Delta_3(r_{12}^{-3} - r_{14}^{-3}) = m_3\Delta_1(r_{23}^{-3} - r_{34}^{-3}), \\ m_1\Delta_2(r_{13}^{-3} - r_{14}^{-3}) = m_2\Delta_1(r_{23}^{-3} - r_{24}^{-3}); \end{array} \tag{1}$$

see Hagihara [3]. Here, r_{ij} is the euclidean distance between the masses m_i and m_j; Δ_1, Δ_2, Δ_3 and Δ_4 denote the oriented areas of the triangles of vertices (m_2, m_3, m_4), (m_4, m_3, m_1), (m_1, m_2, m_4) and (m_3, m_2, m_1), respectively. More precisely,

$$\Delta_1 = \frac{1}{2}\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix},$$

where (x_i, y_i) is the position vector of the mass m_i, and similarly for Δ_2, Δ_3 and Δ_4.

Since we are looking for planar central configurations we can consider a redundant additional equation in system (1) by imposing that the volume of the tetrahedron with vertices the four masses is zero; that is,

$$\begin{vmatrix} 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 & 1 \\ r_{12}^2 & 0 & r_{23}^2 & r_{24}^2 & 1 \\ r_{13}^2 & r_{23}^2 & 0 & r_{34}^2 & 1 \\ r_{14}^2 & r_{24}^2 & r_{34}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0\ ; \tag{2}$$

see [3] for more details.

By Heron's formula, the area of the triangle with edge lengths α, β and γ is given by

$$\sqrt{s(s-\alpha)(s-\beta)(s-\gamma)}\ ,$$

where $s = (\alpha+\beta+\gamma)/2$ is the semiperimeter of the triangle. Therefore Heron's formula allows us to compute $|\Delta_1|$, $|\Delta_2|$, $|\Delta_3|$ and $|\Delta_4|$ as functions of the mutual distances.

For given m_1, m_2, m_3 and m_4, in order to prove the theorem we suppose fixed the positions of the three masses m_1, m_2 and m_3. So, we know the following variables in system (1): $r_{12} = a$, $r_{13} = b$, and $r_{23} = c$. The unknowns are the variables $r_{14} = x$, $r_{24} = y$ and $r_{34} = z$. In short, we can write system (1) in the new notation, obtaining

$$\begin{aligned}
m_3\Delta_4(b^{-3} - c^{-3}) &= m_4\Delta_3(x^{-3} - y^{-3}), && (3)\\
m_2\Delta_4(a^{-3} - c^{-3}) &= m_4\Delta_2(x^{-3} - z^{-3}), && (4)\\
m_2\Delta_3(a^{-3} - y^{-3}) &= m_3\Delta_2(b^{-3} - z^{-3}), && (5)\\
m_1\Delta_4(a^{-3} - b^{-3}) &= m_4\Delta_1(y^{-3} - z^{-3}), && (6)\\
m_1\Delta_3(a^{-3} - x^{-3}) &= m_3\Delta_1(c^{-3} - z^{-3}), && (7)\\
m_1\Delta_2(b^{-3} - x^{-3}) &= m_2\Delta_1(c^{-3} - y^{-3}). && (8)
\end{aligned}$$

Since we have more equations than unknowns, in order to study the finiteness of the solutions of system (3)–(8) we do not need to work with all the equations. In particular, in our study we will only use equations (3), (4), (6), (8) and the redundant equation (2). We replace Δ_1, Δ_2, Δ_3 and Δ_4 in equations (3), (4), (6), (8) by their corresponding expressions given by Heron's formula with the convenient sign. Next we eliminate the square roots that appear in the expression of the Δ's in equations (3), (4), (6), (8) by taking squares. Thus, we get the *polynomial system*

$$f_1 = 0, \qquad f_2 = 0, \qquad f_3 = 0, \qquad f_4 = 0, \tag{9}$$

where

$$\begin{aligned}
f_1 \;=\; & a^4 b^6 c^6 m_4^2 x^6 - 2a^2 b^6 c^6 m_4^2 x^8 + b^6 c^6 m_4^2 x^{10} - 2a^2 b^6 c^6 m_4^2 x^6 y^2 - \\
& 2b^6 c^6 m_4^2 x^8 y^2 - 2a^4 b^6 c^6 m_4^2 x^3 y^3 + 4a^2 b^6 c^6 m_4^2 x^5 y^3 - 2b^6 c^6 m_4^2 x^7 y^3 + \\
& b^6 c^6 m_4^2 x^6 y^4 + 4a^2 b^6 c^6 m_4^2 x^3 y^5 + 4b^6 c^6 m_4^2 x^5 y^5 + a^4 b^6 c^6 m_4^2 y^6 - \\
& 2a^2 b^6 c^6 m_4^2 x^2 y^6 + b^6 c^6 m_4^2 x^4 y^6 - a^4 b^6 m_3^2 x^6 y^6 + 2a^2 b^8 m_3^2 x^6 y^6 - \\
& b^{10} m_3^2 x^6 y^6 + 2a^2 b^6 c^2 m_3^2 x^6 y^6 + 2b^8 c^2 m_3^2 x^6 y^6 + 2a^4 b^3 c^3 m_3^2 x^6 y^6 - \\
& 4a^2 b^5 c^3 m_3^2 x^6 y^6 + 2b^7 c^3 m_3^2 x^6 y^6 - b^6 c^4 m_3^2 x^6 y^6 - 4a^2 b^3 c^5 m_3^2 x^6 y^6 - \\
& 4b^5 c^5 m_3^2 x^6 y^6 - a^4 c^6 m_3^2 x^6 y^6 + 2a^2 b^2 c^6 m_3^2 x^6 y^6 - b^4 c^6 m_3^2 x^6 y^6 + \\
& 2b^3 c^7 m_3^2 x^6 y^6 + 2a^2 c^8 m_3^2 x^6 y^6 + 2b^2 c^8 m_3^2 x^6 y^6 - c^{10} m_3^2 x^6 y^6 - \\
& 2b^6 c^6 m_4^2 x^3 y^7 - 2a^2 b^6 c^6 m_4^2 y^8 - 2b^6 c^6 m_4^2 x^2 y^8 + b^6 c^6 m_4^2 y^{10},
\end{aligned}$$

$$
\begin{aligned}
f_2 \;=\;& a^6\,b^4\,c^6\,m_4^2\,x^6 - 2\,a^6\,b^2\,c^6\,m_4^2\,x^8 + a^6\,c^6\,m_4^2\,x^{10} - 2\,a^6\,b^2\,c^6\,m_4^2\,x^6\,z^2 - \\
& 2\,a^6\,c^6\,m_4^2\,x^8\,z^2 - 2\,a^6\,b^4\,c^6\,m_4^2\,x^3\,z^3 + 4\,a^6\,b^2\,c^6\,m_4^2\,x^5\,z^3 - 2\,a^6\,c^6\,m_4^2\,x^7\,z^3 + \\
& a^6\,c^6\,m_4^2\,x^6\,z^4 + 4\,a^6\,b^2\,c^6\,m_4^2\,x^3\,z^5 + 4\,a^6\,c^6\,m_4^2\,x^5\,z^5 + a^6\,b^4\,c^6\,m_4^2\,z^6 - \\
& 2\,a^6\,b^2\,c^6\,m_4^2\,x^2\,z^6 + a^6\,c^6\,m_4^2\,x^4\,z^6 - a^{10}\,m_2^2\,x^6\,z^6 + 2\,a^8\,b^2\,m_2^2\,x^6\,z^6 - \\
& a^6\,b^4\,m_2^2\,x^6\,z^6 + 2\,a^8\,c^2\,m_2^2\,x^6\,z^6 + 2\,a^6\,b^2\,c^2\,m_2^2\,x^6\,z^6 + 2\,a^7\,c^3\,m_2^2\,x^6\,z^6 - \\
& 4\,a^5\,b^2\,c^3\,m_2^2\,x^6\,z^6 + 2\,a^3\,b^4\,c^3\,m_2^2\,x^6\,z^6 - a^6\,c^4\,m_2^2\,x^6\,z^6 - 4\,a^5\,c^5\,m_2^2\,x^6\,z^6 - \\
& 4\,a^3\,b^2\,c^5\,m_2^2\,x^6\,z^6 - a^4\,c^6\,m_2^2\,x^6\,z^6 + 2\,a^2\,b^2\,c^6\,m_2^2\,x^6\,z^6 - b^4\,c^6\,m_2^2\,x^6\,z^6 + \\
& 2\,a^3\,c^7\,m_2^2\,x^6\,z^6 + 2\,a^2\,c^8\,m_2^2\,x^6\,z^6 + 2\,b^2\,c^8\,m_2^2\,x^6\,z^6 - c^{10}\,m_2^2\,x^6\,z^6 - \\
& 2\,a^6\,c^6\,m_4^2\,x^3\,z^7 - 2\,a^6\,b^2\,c^6\,m_4^2\,z^8 - 2\,a^6\,c^6\,m_4^2\,x^2\,z^8 + a^6\,c^6\,m_4^2\,z^{10},
\end{aligned}
$$

$$
\begin{aligned}
f_3 \;=\;& a^6\,b^6\,c^4\,m_4^2\,y^6 - 2\,a^6\,b^6\,c^2\,m_4^2\,y^8 + a^6\,b^6\,m_4^2\,y^{10} - 2\,a^6\,b^6\,c^2\,m_4^2\,y^6\,z^2 - \\
& 2\,a^6\,b^6\,m_4^2\,y^8\,z^2 - 2\,a^6\,b^6\,c^4\,m_4^2\,y^3\,z^3 + 4\,a^6\,b^6\,c^2\,m_4^2\,y^5\,z^3 - 2\,a^6\,b^6\,m_4^2\,y^7\,z^3 + \\
& a^6\,b^6\,m_4^2\,y^6\,z^4 + 4\,a^6\,b^6\,c^2\,m_4^2\,y^3\,z^5 + 4\,a^6\,b^6\,m_4^2\,y^5\,z^5 + a^6\,b^6\,c^4\,m_4^2\,z^6 - \\
& 2\,a^6\,b^6\,c^2\,m_4^2\,y^2\,z^6 + a^6\,b^6\,m_4^2\,y^4\,z^6 - a^{10}\,m_1^2\,y^6\,z^6 + 2\,a^8\,b^2\,m_1^2\,y^6\,z^6 + \\
& 2\,a^7\,b^3\,m_1^2\,y^6\,z^6 - a^6\,b^4\,m_1^2\,y^6\,z^6 - 4\,a^5\,b^5\,m_1^2\,y^6\,z^6 - a^4\,b^6\,m_1^2\,y^6\,z^6 + \\
& 2\,a^3\,b^7\,m_1^2\,y^6\,z^6 + 2\,a^2\,b^8\,m_1^2\,y^6\,z^6 - b^{10}\,m_1^2\,y^6\,z^6 + 2\,a^8\,c^2\,m_1^2\,y^6\,z^6 + \\
& 2\,a^6\,b^2\,c^2\,m_1^2\,y^6\,z^6 - 4\,a^5\,b^3\,c^2\,m_1^2\,y^6\,z^6 - 4\,a^3\,b^5\,c^2\,m_1^2\,y^6\,z^6 + 2\,a^2\,b^6\,c^2\,m_1^2\,y^6\,z^6 + \\
& 2\,b^8\,c^2\,m_1^2\,y^6\,z^6 - a^6\,c^4\,m_1^2\,y^6\,z^6 + 2\,a^3\,b^3\,c^4\,m_1^2\,y^6\,z^6 - b^6\,c^4\,m_1^2\,y^6\,z^6 - \\
& 2\,a^6\,b^6\,m_4^2\,y^3\,z^7 - 2\,a^6\,b^6\,c^2\,m_4^2\,z^8 - 2\,a^6\,b^6\,m_4^2\,y^2\,z^8 + a^6\,b^6\,m_4^2\,z^{10},
\end{aligned}
$$

$$
\begin{aligned}
f_4 \;=\;& b^6\,c^{10}\,m_2^2\,x^6 - 2\,b^6\,c^8\,m_2^2\,x^6\,y^2 - 2\,b^6\,c^7\,m_2^2\,x^6\,y^3 + b^6\,c^6\,m_2^2\,x^6\,y^4 + \\
& 4\,b^6\,c^5\,m_2^2\,x^6\,y^5 - b^{10}\,c^6\,m_1^2\,y^6 + 2\,b^8\,c^6\,m_1^2\,x^2\,y^6 + 2\,b^7\,c^6\,m_1^2\,x^3\,y^6 - \\
& b^6\,c^6\,m_1^2\,x^4\,y^6 - 4\,b^5\,c^6\,m_1^2\,x^5\,y^6 - b^4\,c^6\,m_1^2\,x^6\,y^6 + b^6\,c^4\,m_2^2\,x^6\,y^6 + \\
& 2\,b^3\,c^6\,m_1^2\,x^7\,y^6 + 2\,b^2\,c^6\,m_1^2\,x^8\,y^6 - c^6\,m_1^2\,x^{10}\,y^6 - 2\,b^6\,c^3\,m_2^2\,x^6\,y^7 - \\
& 2\,b^6\,c^2\,m_2^2\,x^6\,y^8 + b^6\,m_2^2\,x^6\,y^{10} - 2\,b^6\,c^8\,m_2^2\,x^6\,z^2 - 2\,b^6\,c^6\,m_2^2\,x^6\,y^2\,z^2 + \\
& 4\,b^6\,c^5\,m_2^2\,x^6\,y^3\,z^2 + 4\,b^6\,c^3\,m_2^2\,x^6\,y^5\,z^2 + 2\,b^8\,c^6\,m_1^2\,y^6\,z^2 + 2\,b^6\,c^6\,m_1^2\,x^2\,y^6\,z^2 - \\
& 4\,b^5\,c^6\,m_1^2\,x^3\,y^6\,z^2 - 4\,b^3\,c^6\,m_1^2\,x^5\,y^6\,z^2 + 2\,b^2\,c^6\,m_1^2\,x^6\,y^6\,z^2 - 2\,b^6\,c^2\,m_2^2\,x^6\,y^6\,z^2 + \\
& 2\,c^6\,m_1^2\,x^8\,y^6\,z^2 - 2\,b^6\,m_2^2\,x^6\,y^8\,z^2 + b^6\,c^6\,m_2^2\,x^6\,z^4 - 2\,b^6\,c^3\,m_2^2\,x^6\,y^3\,z^4 - \\
& b^6\,c^6\,m_1^2\,y^6\,z^4 + 2\,b^3\,c^6\,m_1^2\,x^3\,y^6\,z^4 - c^6\,m_1^2\,x^6\,y^6\,z^4 + b^6\,m_2^2\,x^6\,y^6\,z^4.
\end{aligned}
$$

Equation (2) becomes

$$
\begin{aligned}
f_5 \;=\;& a^2\,b^2\,c^2 - a^2\,c^2\,x^2 - b^2\,c^2\,x^2 + c^4\,x^2 + c^2\,x^4 - a^2\,b^2\,y^2 + b^4\,y^2 - b^2\,c^2\,y^2 + \\
& a^2\,x^2\,y^2 - b^2\,x^2\,y^2 - c^2\,x^2\,y^2 + b^2\,y^4 + a^4\,z^2 - a^2\,b^2\,z^2 - a^2\,c^2\,z^2 - \\
& a^2\,x^2\,z^2 + b^2\,x^2\,z^2 - c^2\,x^2\,z^2 - a^2\,y^2\,z^2 - b^2\,y^2\,z^2 + c^2\,y^2\,z^2 + a^2\,z^4,
\end{aligned}
$$

after removing the nonzero factor -2.

In short, in order to prove that system (3)–(8) has finitely many solutions, it is sufficient to prove that the polynomial system $f_i = 0$, for $i = 1, \ldots, 5$, also has finitely many solutions.

We will show that system $f_i = 0$, for $i = 1, \ldots, 5$, has finitely many solutions for x, y and z; and we claim that this will imply that the number of possible positions for m_4 is finite for given positions of m_1, m_2 and m_3. Now, we shall prove the claim. We note that knowing x, y and z the position of m_4 must be at the intersection of the three circles centered at m_1, m_2 and m_3 with radii x, y and z, respectively (eventually such intersections can be empty). Therefore, if there are finitely many solutions for x, y and z, then there are finitely many solutions for the position of m_4. So, the claim is proved.

3. Multipolynomial equations

In this section we present a brief summary on the resultant and on the Bézout theorem. Both will be used later on for proving the main theorem.

3.1. The resultant of two polynomials.

Let the roots of the polynomial $P(x)$ with leading coefficient one be denoted by a_i, $i = 1, 2, \ldots, n$ and those of the polynomial $Q(x)$ with leading coefficient one be denoted by b_j, $j = 1, 2, \ldots, m$. The *resultant* of P and Q, $\mathrm{Res}[P, Q]$, is the expression formed by the product of all the differences $a_i - b_j$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. In order to see how to compute $\mathrm{Res}[P, Q]$, see for instance [4] and [9]. The main property of the resultant is that if P and Q have a common solution, then necessarily $\mathrm{Res}[P, Q] = 0$.

Consider now two multivariable polynomials, say $P(X, Y)$ and $Q(X, Y)$. These polynomials can be considered as polynomials in X with polynomial coefficients in Y. Then the resultant with respect to X, $\mathrm{Res}[P, Q, X]$, is a polynomial in the variables Y with the following property. If $P(X, Y)$ and $Q(X, Y)$ have a common solution (X_0, Y_0), then $\mathrm{Res}[P, Q, X](Y_0) = 0$, and similarly for the variable Y. In particular, if the polynomials depending on one variable,

$$\begin{aligned} p(X) &= \mathrm{Res}[P, Q, Y], \\ q(Y) &= \mathrm{Res}[P, Q, X], \end{aligned}$$

have finitely many solutions (i.e. they are not the zero polynomial), then the polynomial system

$$P(X, Y) = 0, \quad Q(X, Y) = 0$$

has finitely many solutions.

3.2. Bézout Theorem.

Let $F_1, \ldots, F_n$ be n homogeneous polynomials of degrees $d_1, \ldots, d_n$ in the variables $x_0, \ldots, x_n$. We define

$$\begin{aligned} f_i(x_1, \ldots, x_n) &= F_i(1, x_1, \ldots, x_n), \\ \overline{F}_i(x_1, \ldots, x_n) &= F_i(0, x_1, \ldots, x_n). \end{aligned} \tag{10}$$

Since the $\overline{F}_i$ are homogeneous polynomials in the variables $x_1, \ldots, x_n$, it is clear that the equations $\overline{F}_i = 0$, for $i = 1, \ldots, n$, always have the solution $x_1 = \cdots = x_n = 0$. This solution is called the *trivial* solution.

Theorem 1 (Bézout Theorem)**.** *Assume that f_i and $\overline{F}_i$ are defined as in* (10)*. If the unique solution of the homogenized system $\overline{F}_i = 0$, for $i = 1, \ldots, n$, is the trivial one, then the system $f_i = 0$, for $i = 1, \ldots, n$, has $d_1 \cdots d_n$ solutions in $\mathbb{C}^n$ (counted with their multiplicity).*

For more details, see [1].

4. The proof

We shall use the following auxiliary result.

Proposition 2. *If three masses are collinear, then there is no position for the remainder mass outside the straight line defined by the collinear three masses in order that they form a central configuration of the 4–body problem.*

Proof. Without loss of generality, we assume that the masses m_1, m_2, m_3 are collinear, and consequently the area of the triangle $\Delta_4 = 0$. Assuming that m_4 is the mass outside the straight line defined by m_1, m_2 and m_3, it follows that $\Delta_1\Delta_2\Delta_3 \neq 0$. Therefore, from system (3)–(8), we get that $x = y = z$, in contradiction with the fact that m_4 is outside the straight line defined by m_1, m_2 and m_3. □

From the proposition it follows that if a central configuration of the 4–body problem is not collinear, then it has no three masses on a straight line. In other words, in the proof of the theorem we can assume that $\Delta_1\Delta_2\Delta_3\Delta_4 \neq 0$.

The proof of the theorem is divided into two cases:

Case 1: We assume that a, b and c are pairwise different.

We consider the following system of equations:

$$(11) \quad g_1 = f_1 + w^{12} = 0, \qquad g_2 = f_2 + w^{12} = 0, \qquad g_3 = f_3 = 0, \qquad g_4 = f_5 = 0,$$

where x, y, z and w are the unknowns. We note that the solutions of system $f_1 = f_2 = f_3 = f_5 = 0$ are solutions of (11) with $w = 0$. Thus, it is easy to see that if system (11) has finitely many solutions (x, y, z, w), then system $f_1 = f_2 = f_3 = f_5 = 0$ also has finitely many solutions (x, y, z). Consequently, the system $f_i = 0$, for $i = 1, \ldots, 5$, will have finitely many solutions (x, y, z). Therefore the main theorem will be proved in Case 1.

In order to see that system (11) has finitely many solutions (x, y, z, w), we will apply the Bézout Theorem. First, we homogenize the system $g_i(x, y, z, w) = 0$, for $i = 1, \ldots, 4$, to the system $G_i(u, x, y, z, w) = 0$, for $i = 1, \ldots, 4$, adding the new variable u in such a way that

$$g_i(x, y, z, w) = G_i(1, x, y, z, w),$$
$$\overline{G}_i(x, y, z, w) = G_i(0, x, y, z, w).$$

By the Bézout Theorem, if the unique solution of the homogenized system $\overline{G}_i = 0$, for $i = 1, \ldots, 4$, is the trivial one, i.e. $x = y = z = w = 0$, then system $g_i = 0$, for $i = 1, \ldots, 4$, has finitely many solutions.

We see that

$$\overline{G}_3 = -(a-b)^2 \left(a^2 + a\,b + b^2\right)^2 (a-b-c)\,(a+b-c)\,(a-b+c)\,(a+b+c)\, m_1^2\, y^6\, z^6.$$

Since we have assumed that $a \neq b$, $a, b, c > 0$ and m_1, m_2 and m_3 are not collinear (i.e. $a - b - c \neq 0$, $a + b - c \neq 0$ and $a - b + c \neq 0$), we have that $\overline{G}_3 = 0$ if and only if either $y = 0$ or $z = 0$.

Assume $y = 0$. Then $\overline{G}_1 = w^{12}$; so $\overline{G}_1 = 0$ implies that $w = 0$. If $y = w = 0$, then

$$\overline{G}_2 = -(a-c)^2\, (a-b-c)\,(a+b-c)\,(a-b+c)\,(a+b+c)\left(a^2 + a\,c + c^2\right)^2 m_2^2\, x^6\, z^6.$$

So, $\overline{G}_2 = 0$ if and only if either $x = 0$ or $z = 0$. Assume that $x = 0$. Then for $x = y = w = 0$ we have that $\overline{G}_4 = a^2\, z^4$. Consequently, $\overline{G}_4 = 0$ if and only if $z = 0$. Hence, in this subcase the unique solution of $\overline{G}_i = 0$, for $i = 1, \ldots, 4$, is the trivial one. On the other hand, for $y = z = w = 0$ we have that $\overline{G}_4 = c^2\, x^4$ and consequently $\overline{G}_4 = 0$ if and only if $x = 0$. Again, the unique solution is the trival one.

Using similar arguments in the case $z = 0$ we can also see that system $\overline{G}_i = 0$, for $i = 1, \ldots, 4$, has a unique solution, the trivial one.

Case 2: We assume now that two of the distances a, b and c are equal. Without loss of generality, we suppose that $c = b$. Since $\Delta_i \neq 0$ for $i = 1, \ldots, 4$ (see Proposition 2), from (3)–(8) it follows that $y = x$.

We consider system

$$h_1(x,z) = f_2|_{\substack{c=b\\ y=x}}, \qquad h_2(x,z) = f_4|_{\substack{c=b\\ y=x}}, \qquad h_3(x,z) = f_5|_{\substack{c=b\\ y=x}}. \tag{12}$$

We want to see that system (12) has finitely many solutions, because this will imply that system $f_i = 0$, for $i = 1, \ldots, 5$, will have finitely many solutions (x, y, z) when $c = b$. After some computations we have that

$$\begin{aligned} h_2 &= -b^6\,(m_1 - m_2)\,(m_1 + m_2)\,(b - x)^2\,x^6\,\left(b^2 + b\,x + x^2\right)^2\,(b - x - z) \\ &\quad \cdot (b + x - z)\,(b - x + z)\,(b + x + z)\,. \end{aligned}$$

Since $b, m_1, m_2 > 0$ and we are looking for solutions of (12) with $x, z > 0$, we have that $h_2 = 0$ if and only if either $m_1 = m_2$, or $x = b$, or $x = b - z$, or $x = z - b$, or $x = b + z$.

We start studying the case $m_1 = m_2$. If $m_1 = m_2$, then $h_1 = -a^2\,\overline{h}_1$, $h_3 = a^2\,\overline{h}_3$, where

$$\begin{aligned} \overline{h}_1 &= -a^4\,b^{10}\,m_4^2\,x^6 + 2\,a^4\,b^8\,m_4^2\,x^8 - a^4\,b^6\,m_4^2\,x^{10} + 2\,a^4\,b^8\,m_4^2\,x^6\,z^2 + \\ &\quad 2\,a^4\,b^6\,m_4^2\,x^8\,z^2 + 2\,a^4\,b^{10}\,m_4^2\,x^3\,z^3 - 4\,a^4\,b^8\,m_4^2\,x^5\,z^3 + 2\,a^4\,b^6\,m_4^2\,x^7\,z^3 - \\ &\quad a^4\,b^6\,m_4^2\,x^6\,z^4 - 4\,a^4\,b^8\,m_4^2\,x^3\,z^5 - 4\,a^4\,b^6\,m_4^2\,x^5\,z^5 - a^4\,b^{10}\,m_4^2\,z^6 + \\ &\quad 2\,a^4\,b^8\,m_4^2\,x^2\,z^6 - a^4\,b^6\,m_4^2\,x^4\,z^6 + a^8\,m_2^2\,x^6\,z^6 - 4\,a^6\,b^2\,m_2^2\,x^6\,z^6 - \\ &\quad 2\,a^5\,b^3\,m_2^2\,x^6\,z^6 + 8\,a^3\,b^5\,m_2^2\,x^6\,z^6 + a^2\,b^6\,m_2^2\,x^6\,z^6 - 4\,b^8\,m_2^2\,x^6\,z^6 + \\ &\quad 2\,a^4\,b^6\,m_4^2\,x^3\,z^7 + 2\,a^4\,b^8\,m_4^2\,z^8 + 2\,a^4\,b^6\,m_4^2\,x^2\,z^8 - a^4\,b^6\,m_4^2\,z^{10} \end{aligned}$$

and

$$\overline{h}_3 = b^4 - 2\,b^2\,x^2 + x^4 + a^2\,z^2 - 2\,b^2\,z^2 - 2\,x^2\,z^2 + z^4.$$

In order to prove that $\overline{h}_1 = \overline{h}_3 = 0$ has finitely many solutions we shall use the resultant (see Section 3 for details). We have that

$$\begin{aligned} \mathrm{Res}[\overline{h}_1, \overline{h}_3, z] &= (b - x)^4\,(b + x)^4\,\left(a^{12}\,b^{24}\,m_4^4 + r_1(x)\right)\,\left(a^{12}\,b^{24}\,m_4^4 + r_2(x)\right), \\ \mathrm{Res}[\overline{h}_1, \overline{h}_3, x] &= z^8\,\left(a^{12}\,b^{24}\,m_4^4 + s_1(z)\right)\,\left(a^{12}\,b^{24}\,m_4^4 + s_2(z)\right), \end{aligned}$$

where $r_1(x)$, $r_2(x)$ and $s_1(z)$, $s_2(z)$ are polynomials of degree 20 in x and z, respectively, without the constant term. Since $a, b, m_4 > 0$, we have that $\mathrm{Res}[\overline{h}_1, \overline{h}_3, z] \not\equiv 0$ and $\mathrm{Res}[\overline{h}_1, \overline{h}_3, x] \not\equiv 0$. Therefore, from Section 3, system $\overline{h}_1 = \overline{h}_3 = 0$ has finitely many solutions, and consequently system (12) has finitely many solutions.

We consider now the case $x = b$. If $x = b$, then $h_3 = -2\,a^2\,z^2\,\left(a^2 - 4\,b^2 + z^2\right)$. Since $h_3 = 0$ has finitely many solutions in the variable z and $x = y = b$, we have that in this case system (12) also has finitely many solutions. Proceeding in a similar way we can see that (12) also has finitely many solutions in the remaining cases $x = b - z$, $x = z - b$ and $x = b + z$. This completes the proof of the theorem.

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